

THE JULIA SET OF A POST-CRITICALLY FINITE ENDOMORPHISM OF $\mathbb{P}\mathbb{C}^2$

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ABSTRACT. We construct a combinatorial model of the Julia set of the endomorphism $f(z, w) = ((1 - 2z/w)^2, (1 - 2/w)^2)$ of $\mathbb{P}\mathbb{C}^2$.

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1. INTRODUCTION

J. E. Fornæss and N. Sibony studied in [FS92] two post-critically finite endomorphisms of $\mathbb{P}\mathbb{C}^2$. The Julia set of one of them is $\mathbb{P}\mathbb{C}^2$, while the Julia set of the other has no interior.

The latter map appeared independently in [BN06] as a natural skew product map in the study of Thurston equivalence of topological polynomials with the post-critical dynamics of $z^2 + i$. It is written in affine coordinates as

$$f(z, w) = \left(\left(1 - \frac{2z}{w}\right)^2, \left(1 - \frac{2}{w}\right)^2 \right).$$

As a development of [BN06], group theoretic aspects of the map f were studied in [Nek07] and [Nek10]. The iterated monodromy group of f was used to construct an uncountable family of three-generated groups with interesting properties, and later a new group of non-uniform exponential growth was found in this family (the first examples of groups of non-uniform exponential growth were found in [Wil04]).

In the current paper we apply our group theoretic knowledge to description of the Julia set of f . We construct a combinatorial model of the Julia set in the spirit of Hubbard trees. Of course, here the Hubbard trees become “Hubbard complexes” (actually even complexes of groups); but due to a particular skew product structure of the map, the Hubbard complex of f is a bundle of “Hubbard tripods”, and the combinatorial model of the Julia set can be described in terms of a “folding map” on the bundle of tripods. The Julia set of f is represented then as the projective limit of a sequences of three-dimensional Hubbard complexes, which are homeomorphic to subsets of the Julia set.

A general method of constructing similar combinatorial models of expanding dynamical systems is described in [Nek08a]. Finding simple and elegant models is still of interest, since construction from [Nek08a] depends on some choices, which can be made in different ways, leading to models of different complexity.

Interesting examples of post-critically finite multi-dimensional maps come from the study of correspondences on the moduli space of a punctured sphere, see [BN06, Koc07, Nek08a].

Unfortunately, an important ingredient of our analysis is missing. I was not able to prove that f is sub-hyperbolic, i.e., to construct a singular metric on a neighborhood of the Julia set of f , such that f is uniformly expanding with respect to it.

1.1. Overview of the paper. The second section of the paper collects elementary and previously known properties of the function f . We describe its action on the projective plane $\mathbb{P}\mathbb{C}^2$, the structure of the post-critical set of f , recall the results of J. E. Fornæss and N. Sibony and discuss the skew product structure of the map.

Section “Techniques” is an overview of the theory of self-similar groups, iterated monodromy groups and their limit spaces. There are no proofs in it, which can be found either in [Nek05], or in [Nek08a]. In particular, the general notion of combinatorial models of expanding dynamical systems is described in this section. More on combinatorial models of hyperbolic dynamical systems, see [IS08] and [Nek08a].

We compute the iterated monodromy group of f in Section 4. We use an interpretation of the map f given in [BN06], which makes it possible to compute the iterated monodromy group $\text{IMG}(f)$ in a relatively easy way.

The combinatorial model of f is constructed in Section 5. It is convenient to pass to an index 2 extension Γ of $\text{IMG}(f)$. This extension can be defined as the iterated monodromy group of the quotient of the dynamical system $(f, \mathbb{P}\mathbb{C}^2)$ by the group of order two generated by the transformation $(z, w) \mapsto (\bar{z}, \bar{w})$. The group Γ is generated by a relatively small automaton (of 12 states). It has appeared for the first time in [Nek07], where it was used to study a Cantor set of groups associated with f . We continue the study of the group Γ in Section 5 of our paper. In particular, we describe its nucleus, which happens to be a union of six finite groups. We use the poset of subgroups of the nucleus to construct a simplicial complex, serving as the first approximation of the limit space of Γ .

This complex consists of three tetrahedra with one common face. The corresponding approximation of the Julia set of f is obtained by pasting together two copies of this complex. We construct the combinatorial model of the Julia set (in Subsection 5.6); give an inductive cut-and-paste rule for constructing a sequences of polyhedra approximating the Julia set (Theorem 5.4 and Proposition 5.8); and prove that Julia set is the inverse limit of the constructed polyhedra (Theorem 5.7).

The cut-and-paste rule works as follows. The n th level approximation \mathcal{M}_n of the Julia set is obtained by pasting together two copies of a complex \mathcal{T}_n along their “boundary”. The boundary of \mathcal{T}_n is decomposed into a union of 11 regions, which are domains of involutive maps $\kappa_{g,n}$ (“pasting rules”). The next complex \mathcal{T}_{n+1} is obtained by pasting four copies of \mathcal{T}_n using five of the maps $\kappa_{g,n}$. The 11 pieces of the boundary of \mathcal{T}_{n+1} and the corresponding maps $\kappa_{g,n+1}$ are defined then as unions of the pieces of the boundaries of the copies of \mathcal{T}_n according to rules described by the finite automaton generating the group Γ .

The rules are not very complicated, but since the complexes are three dimensional and can not be embedded into \mathbb{R}^3 without self-intersections, it is hard to visualize them.

In order to understand better the constructed polyhedral model of the Julia set, we use the skew product structure of f (and of the model), and study it “fiberwise” in Section 6. We show that the fibers of the complexes \mathcal{M}_n are trees that can be constructed using natural “folding” and “unfolding” transformation (Theorem 6.2 and Proposition 6.6). As a limit of iterations of the folding and unfolding procedures we get dendrites homeomorphic to the intersections of the Julia set of f with the lines $w = w_0$.

Our models are very similar to the usual Hubbard trees, since the approximating complexes \mathcal{M}_n are homeomorphic to the subsets $f^{-n}(\mathcal{M})$, where \mathcal{M} is a “span” of the post-critical set of f inside the Julia set of f (Proposition 6.8).

The last section of our paper contains additional results deduced from the model of the Julia set and from properties of the iterated monodromy group of f . We construct a family of length metrics on the slices of the Julia set of f ; define a family of natural surjections of the slices onto an isosceles right triangle (this includes, for instance, the Sierpiński plane-filling curve as a particular case); describe when the slices are finite trees; and describe the action of f on the manifold of “external rays” of the Julia set of f . It is shown that the manifold of external rays is an orbispace with the universal covering identified with the real Heisenberg group, so that the

action of f is induced by an expanding automorphism of the Heisenberg group. Note that in the classical case of polynomials of degree d the space of external rays is \mathbb{R}/\mathbb{Z} with the action of the polynomial induced by the automorphism $x \mapsto d \cdot x$ of \mathbb{R} .

2. THE RATIONAL FUNCTION

Consider the transformation of \mathbb{C}^2

$$f(z, w) = \left(\left(1 - \frac{2z}{w}\right)^2, \left(1 - \frac{2}{w}\right)^2 \right).$$

It can be extended to a map $f : \mathbb{P}\mathbb{C}^2 \rightarrow \mathbb{P}\mathbb{C}^2$ as

$$f : [z : w : u] \mapsto [(w - 2z)^2 : (w - 2u)^2 : w^2].$$

The Jacobian of the map f is then

$$\begin{vmatrix} -4(w - 2z) & 0 & 0 \\ 2(w - 2z) & 2(w - 2u) & 2w \\ 0 & -4(w - 2u) & 0 \end{vmatrix} = -32(w - 2z)(w - 2u)w,$$

Hence, the critical locus is the union of the lines $w = 2z$, $w = 2u$, and $w = 0$. Their orbits are

$$\{w = 2z\} \mapsto \{z = 0\} \mapsto \{z = u\} \mapsto \{z = w\} \mapsto \{z = u\}$$

and

$$\{w = 2u\} \mapsto \{w = 0\} \mapsto \{u = 0\} \mapsto \{w = u\} \mapsto \{w = u\}.$$

Consequently, the post-critical set is the union of six lines

$$z = 0, \quad z = u, \quad z = w, \quad w = 0, \quad w = u, \quad u = 0,$$

or, in affine coordinates: $z = 0$, $z = 1$, $z = w$, $w = 0$, $w = 1$, and the line at infinity.

The function f appeared in [BN06], where it was used to answer a question of J. Hubbard and A. Duady from [DH93] on combinatorial equivalence of some branched coverings of the plane. Groups associated with it were studied in [Nek07] and [Nek10].

This function is conjugate to the function

$$\tilde{f}([z : w : t]) = [(z - 2w)^2 : z^2 : (z - 2t)^2],$$

considered by J. E. Fornæss and N. Sibony in [FS92]. The conjugating map is

$$z \mapsto w, \quad w \mapsto u, \quad t \mapsto z,$$

where the variables on the left-hand side are from [FS92], while the variables from the right-hand side are the ones used in our paper.

The following properties of the map f are proved in [FS92]. Denote by V the post-critical set of f and by W its full preimage $f^{-1}(V)$.

Theorem 2.1. *The sets $\mathbb{P}\mathbb{C}^2 \setminus V$ and $\mathbb{P}\mathbb{C}^2 \setminus W$ are Kobayashi hyperbolic and the map $f : \mathbb{P}\mathbb{C}^2 \setminus W \rightarrow \mathbb{P}\mathbb{C}^2 \setminus V$ is noncontracting in the infinitesimal Kobayashi metric on $\mathbb{P}\mathbb{C}^2 \setminus V$.*

The point $[1 : 0 : 0]$ is a superattracting fixed point. Let U be its basin of attraction and let $J = \mathbb{P}\mathbb{C}^2 \setminus U$ be its complement. Then U is a topological cell and a Kobayashi hyperbolic domain of holomorphy.

The set J has no interior and is the Julia set of f . The map f is topologically transitive on J and repelling periodic points are dense in J .

The aim of our paper is to describe the combinatorics and topology of the Julia set J in the spirit of Hubbard trees (see [DH84, DH85, Poi93]).

Theorem 2.1 is not quite what we need to be able to apply the techniques of the iterated monodromy groups to the study of the Julia set of f . The results of our paper are therefore true only modulo the following conjecture.

Conjecture. *There exists an orbispace metric on $\mathbb{P}\mathbb{C}^2 \setminus \{[1 : 0 : 0]\}$ such that f is uniformly expanding with respect to this metric on a neighborhood of the Julia set.*

In fact, some weaker results would be sufficient, but they are probably equivalent to the above conjecture. It would be very nice to have a general statement about sub-hyperbolicity of post-critically finite endomorphisms of complex projective spaces.

Note that the map f has a skew product structure: the second coordinate is a rational function depending only on the second coordinate. On the first coordinates of iterations of f we get compositions of quadratic polynomials $f_w(z) = (1 - \frac{2}{w}z)^2$, i.e., non-autonomous iteration of quadratic polynomials. The critical point $z = w/2$ of the polynomial f_w is mapped to 0, $f_w(0) = 1$, and $f_w(1)$ is equal to the next value $(1 - \frac{2}{w})^2$ of the second coordinate in the iteration of f . We see that the non-autonomous iteration on the first coordinate is post-critically finite: the set of critical values of the composition $f_{w_n} \circ f_{w_{n-1}} \circ \dots \circ f_{w_1}$ belongs to the set $\{0, 1, w_{n+1}\}$, where $w_{i+1} = (1 - \frac{2}{w_i})^2$ for $i = 1, \dots, n$. For more on post-critically finite non-autonomous iterations of polynomials, see [Nek09].

It follows from the skew-product structure of the map f that it agrees with the projection $P : [z : w : u] \mapsto [w : u]$ (defined on the complement of the point $[1 : 0 : 0]$), which is written in affine coordinates as $P : (z, w) \mapsto w$. Namely, the fibers of the projection are mapped by f to fibers. In particular, by Theorem 2.1, the fiber $P^{-1}(w) \cap J$ of the projection $P : J \rightarrow \widehat{\mathbb{C}}$ of the Julia set of f onto the sphere (which is the Julia set of $(1 - \frac{2}{w})^2$) is the Julia set of the non-autonomous iteration

$$\mathbb{C} \xrightarrow{f_{w_0}} \mathbb{C} \xrightarrow{f_{w_1}} \mathbb{C} \xrightarrow{f_{w_2}} \dots,$$

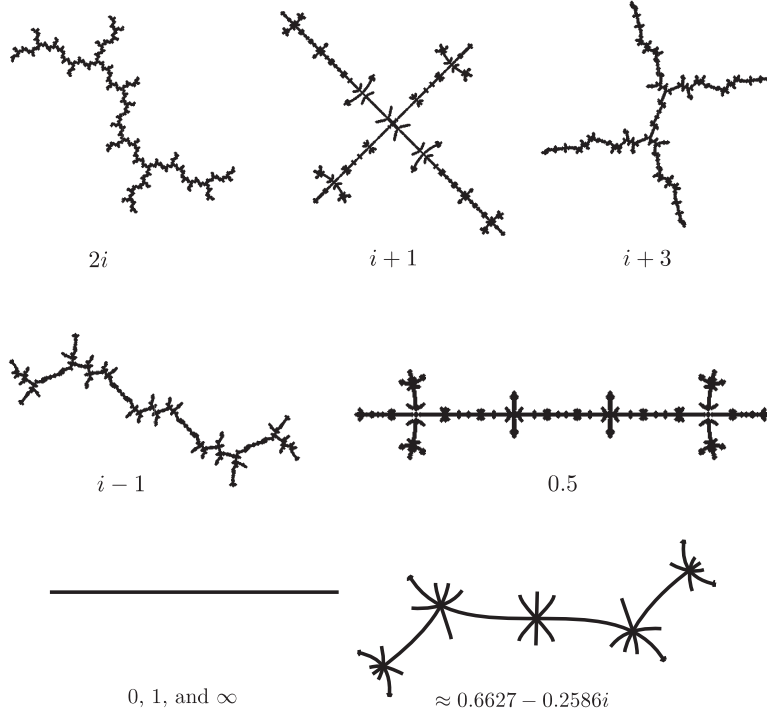
where $w_0 = w$ and $w_{i+1} = (1 - \frac{2}{w_i})^2$.

This makes it possible to draw the slices $P^{-1}(w) \cap J$ of the Julia set of f in z -planes. See some of such slices on Figure 1.

The rational function appearing on the second coordinate of f is a Lattès example. Namely, it is semiconjugate to the map $z \mapsto (i - 1)z$ on \mathbb{C} , where the semiconjugacy is the map

$$\mathbb{C} \rightarrow \mathbb{P}\mathbb{C}^1 : z \mapsto (\wp(z))^2,$$

where \wp is the Weierstrass' function associated with the lattice of Gaussian integers $\mathbb{Z}[i]$. See a proof of this fact in [Mil04] and [BN06]. In other words, the rational function $(1 - \frac{2}{w})^2$ is conjugate to the map induced by $z \mapsto (i - 1)z$ on the orbifold of the action of the group of orientation-preserving isometries of the lattice $\mathbb{Z}[i]$ of Gaussian integers.

FIGURE 1. Intersections of the Julia set of f with the z -planes

3. TECHNIQUES

3.1. Self-similar groups.

Definition 3.1. A *wreath recursion* is a homomorphism $\psi : G \rightarrow \mathfrak{S}(X) \wr G$ from a group G to the wreath product of G with the symmetric group $\mathfrak{S}(X)$.

Here symmetric group $\mathfrak{S}(X)$ acts on X from the left. We will denote the identity element of a group by ε .

The wreath product $\mathfrak{S}(X) \wr G = \mathfrak{S}(X) \ltimes G^X$ is the set of pairs $(\sigma, (g_x)_{x \in X})$, where $\sigma \in \mathfrak{S}(X)$ and $(g_x)_{x \in X}$ is an element of the direct product G^X . The elements of the wreath product are multiplied by the rule

$$(\sigma, (g_x)_{x \in X}) \cdot (\pi, (h_x)_{x \in X}) = (\sigma\pi, (g_{\pi(x)}h_x)_{x \in X}).$$

We will write the pair $(\sigma, (g_x)_{x \in X})$ as a product $\sigma(g_x)_{x \in X}$, identifying σ with $(\sigma, (\varepsilon)_{x \in X})$ and $(g_x)_{x \in X}$ with $(\varepsilon, (g_x)_{x \in X})$. It is easy to see that this identification agrees with the multiplication rule. If $X = \{1, 2, \dots, d\}$, then we write $\sigma(g_x)_{x \in X}$ as a sequence $\sigma(g_1, g_2, \dots, g_d)$.

Let $\psi : G \rightarrow \mathfrak{S}(X) \wr G$ be a wreath recursion. Then the *associated permutation G -bimodule* \mathfrak{M} is the set $X \times G$ together with two (left and right) actions of G on it that are given by the rules

$$(x, h) \cdot g = (x, hg), \quad g \cdot (x, h) = (\sigma(x), g_x h),$$

for $\psi(g) = \sigma(g_x)_{x \in X}$. We will identify $x \in X$ with $(x, \varepsilon) \in \mathfrak{M}$ and write $(x, g) = x \cdot g$.

If we compose a wreath recursion $\psi : G \rightarrow \mathfrak{S}(X) \wr G$ with an inner automorphism of $\mathfrak{S}(X) \wr G$, then we do not change the isomorphism class of the associated bimodule (see [Nek05, Proposition 2.3.4] or [Nek08b, Proposition 2.22]).

Definition 3.2. A *self-similar group* (G, X) is a group G together with a wreath recursion $\psi : G \rightarrow \mathfrak{S}(X) \wr G$. Self-similar groups defined by wreath recursions that differ by an inner automorphism of $\mathfrak{S}(X) \wr G$ are called *equivalent*.

Equivalently, a self-similar group is a group G together with a *covering bimodule* \mathfrak{M} , as it is defined below.

Definition 3.3. Let G be a group. A *permutational G -bimodule* is a set \mathfrak{M} with commuting left and right actions of G on it, i.e., maps $G \times \mathfrak{M} \rightarrow \mathfrak{M} : (g, x) \mapsto g \cdot x$ and $\mathfrak{M} \times G \rightarrow \mathfrak{M} : (x, g) \mapsto x \cdot g$ such that $\varepsilon \cdot x = x \cdot \varepsilon = x$ for all $x \in \mathfrak{M}$ and

$$\begin{aligned} g_1 \cdot (g_2 \cdot x) &= g_1 g_2 \cdot x, & (x \cdot g_1) \cdot g_2 &= x \cdot g_1 g_2, \\ (g_1 \cdot x) \cdot g_1 &= g_1 \cdot (x \cdot g_2), \end{aligned}$$

for all $g_1, g_2 \in G$ and $x \in \mathfrak{M}$.

A permutational G -bimodule is called a *covering d -fold* bimodule if the right action of G on \mathfrak{M} has d orbits and is free, i.e., if $x \cdot g = x$ implies $g = \varepsilon$.

It is easy to see that the bimodule associated with a wreath recursion $\psi : G \rightarrow \mathfrak{S}(X) \wr G$ is a covering d -fold bimodule for $d = |X|$. In the other direction, if \mathfrak{M} is a covering G -bimodule, then for a given right orbit transversal X (i.e., such a subset $X \subset \mathfrak{M}$ that every right orbit contains exactly one element of X), we get the associated wreath recursion

$$g \mapsto \sigma(g|_x)_{x \in X},$$

where $\sigma \in \mathfrak{S}(X)$ and $g|_x$ are given by the condition

$$g \cdot x = \sigma(x) \cdot g|_x$$

in \mathfrak{M} . The permutation σ is the associated action of g on X . We get in this way an action of G on X defined by the wreath recursion. More formally, this action is obtained by composing the wreath recursion with the projection of $\mathfrak{S}(X) \wr G$ onto $\mathfrak{S}(X)$. We will usually denote $g(x) = \sigma(x)$.

We say that a subset $X \subset \mathfrak{M}$ is a *basis* of the bimodule \mathfrak{M} , if it is a right orbit transversal. We will usually label the letters of X by integers $1, 2, \dots, d$, thus identifying $\mathfrak{S}(X)$ with $\mathfrak{S}(d)$. It is not hard to prove that changing the basis X amounts to composing the associated wreath recursion by an inner automorphism of $\mathfrak{S}(d) \wr G$ (see [Nek05, Subsection 2.3.4] and [Nek08b, Proposition 2.22]).

The following result is proved in [Nek05, Section 2.5].

Proposition 3.1. *Let G be a self-similar group with the associated wreath recursion $\psi : G \rightarrow \mathfrak{S}(X) \wr G$. Suppose that the associated action on X is transitive. Then the wreath recursion, up to composition with an inner automorphism of $\mathfrak{S}(X) \wr G$, is uniquely determined for any $x \in X$ by the homomorphism*

$$g \mapsto g_x,$$

from the stabilizer of x into G . Here g_x is the coordinate of $\psi(g) = \sigma(g_x)_{x \in X}$ corresponding to x .

The homomorphism $g \mapsto g_x$ defined on the stabilizer of x is called the *virtual endomorphism* associated with the wreath recursion (or with self-similarity of the group).

The following proposition gives formulae for the wreath recursion with a given associated virtual endomorphism.

Proposition 3.2. *Let $\phi : G_1 \rightarrow G$ be a virtual endomorphism of a group G , where G_1 is a subgroup of finite index in G . Let $\{r_1, r_2, \dots, r_d\}$ be a left coset transversal of G modulo G_1 (i.e., a set such that G is a disjoint union of the cosets $r_i \cdot G_1$). Let $\{x_i\}_{i=1, \dots, d} = \mathbf{X}$ be an alphabet of size d . For $g \in G$ put $\psi(g) = \sigma(g_x)_{x \in \mathbf{X}}$, where $\sigma(x_i) = x_j$ if $gr_i \in r_j G_1$, and $g_{x_i} = \phi(r_j^{-1} gr_i)$. Then $\psi : G \rightarrow \mathfrak{S}(\mathbf{X}) \wr G$ is a wreath recursion such that ϕ is associated with it.*

3.2. Iteration of the wreath recursion. Let (G, \mathbf{X}) be a self-similar group and let $\mathfrak{M} = \mathbf{X} \times G$ be the associated permutational bimodule. For $x \in \mathbf{X}$ and $g \in G$ we denote

$$g \cdot x = g(x) \cdot g|_x.$$

If ψ is the associated wreath recursion, then for $\psi(g) = \sigma(g_x)_{x \in \mathbf{X}}$ we have $g(x) = \sigma(x)$ and $g|_x = g_x$.

We define then inductively, for a finite word $v = x_1 x_2 \dots x_n \in \mathbf{X}^*$ and $g \in G$, a word $g(v)$ and an element $g|_v \in G$ by the rule

$$g(xv) = g(x)g|_x(v), \quad g|_{xv} = g|_x|_v.$$

It is easy to see that for every $n \geq 1$ the map $\sigma_{n,g} : v \mapsto g(v)$ is a permutation of the set \mathbf{X}^n and that the map

$$\psi^{\otimes n} : G \rightarrow \mathfrak{S}(\mathbf{X}^n) \wr G : g \mapsto \sigma_{n,g}(g|_v)_{v \in \mathbf{X}^n}$$

is a homomorphism. The wreath recursion $\psi^{\otimes n}$ is called the *n th iteration* of the wreath recursion ψ .

Let $\mathbf{X}^* = \bigsqcup_{n \geq 0} \mathbf{X}^n$ be the rooted tree of finite words over \mathbf{X} , where every word $v \in \mathbf{X}^*$ is connected to the words of the form vx for $x \in \mathbf{X}$. The empty word is the root of the tree \mathbf{X}^* . It is easy to check that for every $g \in G$ the permutation $v \mapsto g(v)$ of \mathbf{X}^* is an automorphism of the rooted tree \mathbf{X}^* , and that in this way we get an action of the group G on the tree \mathbf{X}^* . It is called the *action associated with the bimodule (with the wreath recursion)*. This action, up to conjugacy of actions, depends only on the associated permutational bimodule (does not depend on the choice of the basis \mathbf{X}).

Definition 3.4. Let $\psi : G \rightarrow \mathfrak{S}(\mathbf{X}) \wr G$ be a wreath recursion. A *faithful quotient* of G (with respect to ψ) is the quotient of G by the kernel of the action on \mathbf{X}^* associated with the wreath recursion ψ .

The wreath recursion is interpreted then as a recurrent description of the action of the group elements on the tree \mathbf{X}^* . For $g \in G$ such that $\psi(g) = \sigma(g|_x)_{x \in \mathbf{X}}$, the permutation $\sigma \in \mathfrak{S}(\mathbf{X})$ describes the action of g on the first level \mathbf{X} of the tree \mathbf{X}^* , while the coordinates $g|_x$ describe the action on the subtree $x\mathbf{X}^*$, so that

$$g(xv) = \sigma(x)g|_x(v)$$

for all $v \in \mathbf{X}^*$. In general, we have

$$g(vw) = g(v)g|_v(w)$$

for all $v, w \in X^*$ and $g \in G$. Note that if the action of G on X^* is faithful, then the above equality uniquely determines $g|_v$. The elements $g|_v$ are called the *sections* of the element $g \in G$.

If the action is faithful, then we identify the elements of the group G with the corresponding automorphisms of the rooted tree X^* . We will usually omit then the letter denoting the wreath recursion and write $g = \sigma(g_1, \dots, g_d)$ instead of $\psi(g) = \sigma(g_1, \dots, g_d)$, naturally identifying the automorphism group $\text{Aut}(X^*)$ with the wreath product $\mathfrak{S}(X) \wr \text{Aut}(X^*)$.

Iterations of wreath recursions correspond to tensor powers of the associated bimodule.

Definition 3.5. Let \mathfrak{M}_1 and \mathfrak{M}_2 be permutational G -bimodules (i.e., sets with commuting left and right actions of G). Then their *tensor product* $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ is the quotient of the direct product $\mathfrak{M}_1 \times \mathfrak{M}_2$ by the identifications

$$x_1 \cdot g \otimes x_2 = x_1 \otimes g \cdot x_2$$

together with the actions

$$g_1 \cdot (x_1 \otimes x_2) \cdot g_2 = (g_1 \cdot x_1) \otimes (x_2 \cdot g_2).$$

If \mathfrak{M}_2 is a set with a right (resp. left) action of G and \mathfrak{M}_1 is a G -bimodule, then the right G -space $\mathfrak{M}_1 \otimes \mathfrak{M}_2$ (resp. left G -space $\mathfrak{M}_2 \otimes \mathfrak{M}_1$) are defined in a similar way.

One can show that the bimodule associated with the n th iteration $\psi^{\otimes n}$ of a wreath recursion ψ is isomorphic to the n th tensor power $\mathfrak{M}^{\otimes n}$ of the bimodule \mathfrak{M} associated with ψ .

If the associated action on X^* is *level-transitive* (i.e., transitive on the levels X^n of X^*), then the virtual endomorphism associated with the n th iterate $\psi^{\otimes n}$ of the wreath recursion is conjugate (i.e., is equal, up to inner automorphisms of the group) to the n th iterate of the virtual endomorphism associated with ψ .

Definition 3.6. Let (G, X) be a self-similar group. A subset $A \subset G$ is said to be *state-closed* (or *self-similar*) if for every $g \in A$ and $x \in X$ we have $g|_x \in A$.

If A is a state-closed subset of G , then it can be considered as an *automaton*, which being in a state $g \in A$ and reading a letter $x \in X$ on input, gives on output the letter $g(x)$ and changes its internal state to $g|_x$. It is easy to see that if it processes a word $v \in X^*$ in this way, then it will give on output the word $g(v)$.

Definition 3.7. Let $A \subset G$ be a state-closed subset of a self-similar group (G, X) . Then its *Moore diagram* is the oriented graph with the set of vertices A , where for every $x \in X$ and $g \in A$ there is an arrow starting in g , ending in $g|_x$, and labeled by x . Every vertex g of the Moore diagram is labeled by the permutation $x \mapsto g(x)$ of X .

3.3. Contracting self-similar groups and their limit spaces.

Definition 3.8. A self-similar group (G, X) is called *contracting* if there exists a finite set $\mathcal{N} \subset G$ such that for every $g \in G$ there exists n_0 such that for every $v \in X^n$ for $n \geq n_0$ we have $g|_v \in \mathcal{N}$. The smallest set \mathcal{N} satisfying this condition is called the *nucleus* of the group (G, X) .

Let us fix some contracting self-similar group (G, X) . Denote by $X^{-\omega}$ the space of sequences $\dots x_2 x_1$, $x_i \in X$, with the direct product topology. By $X^{-\omega} \times G$ we denote the direct product of the space $X^{-\omega}$ with the discrete group G . We write the elements of $X^{-\omega} \times G$ in the form $\dots x_2 x_1 \cdot g$ for $x_i \in X$ and $g \in G$.

Definition 3.9. Two sequences $\dots x_2 x_1, \dots y_2 y_1 \in X^{-\omega}$ are said to be *asymptotically equivalent* (with respect to (G, X)) if there exists a sequence $g_n \in G$ taking values in a finite subset of G such that

$$g_n(x_n \dots x_2 x_1) = y_n \dots y_2 y_1,$$

for all $n \geq 1$. Two sequences $\dots x_2 x_1 \cdot g, \dots y_2 y_1 \cdot h \in X^{-\omega} \times G$ are *asymptotically equivalent* if there exists a sequence $g_n \in G$ taking values in a finite subset of G such that

$$g_n \cdot x_n \dots x_2 x_1 \cdot g = y_n \dots y_2 y_1 \cdot h,$$

in $\mathfrak{M}^{\otimes n}$ for all $n \geq 1$.

Here $x_n \dots x_2 x_1$ denotes the element $x_n \otimes \dots \otimes x_1 \otimes x_1$ of $\mathfrak{M}^{\otimes n}$. Note that the last equality in the definition is equivalent to the conditions

$$g_n(x_n \dots x_2 x_1) = y_n \dots y_2 y_1, \quad g_n|_{x_n \dots x_2 x_1} \cdot g = h,$$

for the n th iteration of the associated wreath recursion.

The following description of the asymptotic equivalence relations is proved in [Nek05, Proposition 3.2.6 and Theorem 3.6.3].

Proposition 3.3. *Sequences $\dots x_2 x_1, \dots y_2 y_1 \in X^{-\omega}$ are asymptotically equivalent if and only if there exists a sequence g_n , $n \geq 0$, of elements of the nucleus of G such that $g_n \cdot x_n = y_n \cdot g_{n-1}$ for all $n \geq 1$.*

Sequences $\dots x_2 x_1 \cdot g, \dots y_2 y_1 \cdot h \in X^{-\omega} \times G$ are asymptotically equivalent if and only if there exists a sequence g_n , $n \geq 0$, of elements of the nucleus of G such that $g_n \cdot x_n = y_n \cdot g_{n-1}$ for all $n \geq 1$, and $g_0 g = h$.

Definition 3.10. The quotient of the space $X^{-\omega}$ by the asymptotic equivalence relation is called the *limit space* of the group (G, X) and is denoted \mathcal{J}_G . The quotient of $X^{-\omega} \times G$ by the asymptotic equivalence relation is called the *limit G -space* and is denoted \mathcal{X}_G .

The asymptotic equivalence relations on $X^{-\omega}$ and $X^{-\omega} \times G$ are invariant with respect to the shift $\dots x_2 x_1 \mapsto \dots x_3 x_2$ and the right G -action $g : \dots x_2 x_1 \cdot h \mapsto \dots x_2 x_1 \cdot (hg)$, respectively. Hence we get a continuous map $s : \mathcal{J}_G \rightarrow \mathcal{J}_G$ induced by the shift, and a natural right action of G on \mathcal{X}_G . The space of orbits \mathcal{X}_G/G of the action is naturally homeomorphic to \mathcal{J}_G .

For every element $x \cdot g$ of the bimodule associated with (G, X) we have a continuous map $\xi \mapsto \xi \otimes x \cdot g$ mapping a point ξ represented by a sequence $\dots x_2 x_1 \cdot h$ to the point represented by

$$\dots x_2 x_1 h(x) \cdot h|_x g.$$

Recall that $h \cdot x \cdot g = h(x) \cdot h|_x g$ in the bimodule $X \cdot G$ associated with (G, X) .

For more on contracting groups and their limit spaces, in particular for examples, see [Nek05, Section 2.11 and Chapter 6].

3.4. Approximation of \mathcal{X}_G by G -spaces. For more on the subject of this subsection (in particular for proofs) see [Nek08a].

Let (G, X) be a self-similar contracting group with the associated wreath recursion $\psi : G \rightarrow \mathfrak{S}(X) \wr G$ and the permutational bimodule $\mathfrak{M} = X \cdot G$.

If \mathcal{X} is a topological space on which G acts from the right side by homeomorphisms, then we denote by $\mathcal{X} \otimes \mathfrak{M}$ the quotient of the direct product of the topological spaces $\mathcal{X} \times \mathfrak{M}$ (where \mathfrak{M} is discrete) by the identifications

$$\xi \cdot g \otimes x = \xi \otimes g \cdot x$$

for $\xi \in \mathcal{X}$, $g \in G$, and $x \in \mathfrak{M}$. It is a right G -space with respect to the action

$$(\xi \otimes x) \cdot g = \xi \otimes (x \cdot g).$$

A map $I : \mathcal{X} \otimes \mathfrak{M} \rightarrow \mathcal{X}$ is said to be *equivariant* if $I(\xi \otimes x \cdot g) = I(\xi \otimes x) \cdot g$ for all $\xi \otimes x \in \mathcal{X} \otimes \mathfrak{M}$ and $g \in G$.

For example, consider the limit G -space \mathcal{X}_G . Then there is a canonical equivariant homeomorphism between $\mathcal{X}_G \otimes \mathfrak{M}$ and \mathcal{X}_G induced by the map

$$X^{-\omega} \times G \times \mathfrak{M} \rightarrow X^{-\omega} \times G : (\dots x_2 x_1 \cdot h, x \cdot g) \mapsto \dots x_2 x_1 h(x) \cdot (h|_x g)$$

already mentioned above (see also [Nek05, Section 3.4]).

If $I : \mathcal{X} \otimes \mathfrak{M} \rightarrow \mathcal{X}$ is a G -equivariant map, then we denote by $I^{(n)}$ the map from $\mathcal{X} \otimes \mathfrak{M}^{\otimes n}$ to \mathcal{X} given by

$$I^{(n)}(\xi \otimes x_1 \otimes x_2 \otimes \dots \otimes x_n) = I(\dots I(I(\xi \otimes x_1) \otimes x_2) \dots \otimes x_n),$$

for $x_i \in \mathfrak{M}$, and by $I_n : \mathcal{X} \otimes \mathfrak{M}^{\otimes(n+1)} \rightarrow \mathcal{X} \otimes \mathfrak{M}^{\otimes n}$ the map given by

$$I_n(\xi \otimes x \otimes v) = I(\xi \otimes x) \otimes v$$

for $v \in \mathfrak{M}^{\otimes n}$ and $x \in \mathfrak{M}$. It is not hard to see that $I^{(n)}$ and I_n are G -equivariant.

Definition 3.11. Suppose that (G, X) is a self-similar group, and let G act on the metric space (\mathcal{X}, d) by isometries properly and co-compactly. An equivariant map $I : \mathcal{X} \otimes \mathfrak{M} \rightarrow \mathcal{X}$ is *contracting* if there exists an integer $n \geq 1$ and a number $0 < \lambda < 1$ such that

$$d(I^{(n)}(\xi_1 \otimes v), I^{(n)}(\xi_2 \otimes v)) \leq \lambda d(\xi_1, \xi_2),$$

for all $\xi_1, \xi_2 \in \mathcal{X}$ and $v \in \mathfrak{M}^{\otimes n}$.

An action of G on \mathcal{X} is said to be *proper* if for every compact subset $C \subset \mathcal{X}$ the set of elements $g \in G$ such that $C \cdot g \cap C \neq \emptyset$ is finite. It is called *co-compact* if there exists a compact subset $C \subset \mathcal{X}$ such that every G -orbit contains a point in C .

If there exists a contracting equivariant map $I : \mathcal{X} \otimes \mathfrak{M} \rightarrow \mathcal{X}$, then $\mathcal{X} \otimes \mathfrak{M}^{\otimes n}$ are approximations of the limit G -space \mathcal{X}_G in the following sense.

Theorem 3.4. *Let (G, X) be a contracting group and let \mathfrak{M} be the associated permutational G -bimodule. Suppose that \mathcal{X} is a locally compact metric space with a co-compact proper right G -action by isometries, and let $I : \mathcal{X} \otimes \mathfrak{M} \rightarrow \mathcal{X}$ be a contracting equivariant map. Then the inverse limit of the G -spaces and the G -equivariant maps*

$$\mathcal{X} \xleftarrow{I_1} \mathcal{X} \otimes \mathfrak{M} \xleftarrow{I_2} \mathcal{X} \otimes \mathfrak{M}^{\otimes 2} \xleftarrow{I_3} \mathcal{X} \otimes \mathfrak{M}^{\otimes 3} \xleftarrow{I_4} \dots,$$

is homeomorphic as a G -space to the limit G -space \mathcal{X}_G (i.e., there exists an equivariant homeomorphism between the inverse limit and \mathcal{X}_G).

It follows that, in the setting of the previous theorem, the orbispaces $\mathcal{M}_n = \mathcal{X} \otimes \mathfrak{M}^{\otimes n}/G$ are approximations of the limit space \mathcal{J}_G . More precisely we have the following.

Corollary 3.5. *In conditions of Theorem 3.4 the limit space \mathcal{J}_G is homeomorphic to the inverse limit of the quotients $\mathcal{M}_n = \mathcal{X} \otimes \mathfrak{M}^{\otimes n}/G$ with respect to the maps $\iota_n : \mathcal{M}_{n+1} \rightarrow \mathcal{M}_n$ induced by I_n .*

The shift map $s : \mathcal{J}_G \rightarrow \mathcal{J}_G$ is the limit of the maps $p_n : \mathcal{M}_{n+1} \rightarrow \mathcal{M}_n$ induced by the correspondence $\xi \otimes x_1 \otimes \cdots \otimes x_n \mapsto \xi \otimes x_1 \otimes \cdots \otimes x_{n-1}$.

3.5. Polyhedral models of the limit space. There is a standard procedure of constructing a G -space satisfying conditions of Theorem 3.4.

Let (G, X) be a contracting finitely generated group. Suppose that it is also *self-replicating* (or *recurrent* in terms of [Nek05]), i.e., that it is transitive on the first level of the tree X^* and the associated virtual endomorphism is onto. Let \mathcal{N} be the nucleus of (G, X) . It is a generating set of G , by [Nek05, Proposition 2.11.3].

Denote by $\overline{\Xi}$ Cayley-Rips complex of G with respect to the generating set \mathcal{N} , i.e., the simplicial complex with the set of vertices G in which a subset $A \subset G$ is a simplex if $A \cdot g^{-1} \subset \mathcal{N}$ for all $g \in A$. The action of G on itself by right translations is simplicial on $\overline{\Xi}$, and we get in this way a right proper co-compact G -space $\overline{\Xi}$.

The map $I : \overline{\Xi} \otimes \mathfrak{M} \rightarrow \overline{\Xi}$ given by the rule

$$I(g \otimes x \cdot h) = g|_x h$$

is a well defined and G -equivariant continuous map. Its iteration $I^{(n)} : \overline{\Xi} \otimes \mathfrak{M}^{\otimes n} \rightarrow \overline{\Xi}$ is defined by

$$I(g \otimes v \cdot h) = g|_v h$$

for $g, h \in G$ and $v \in X^n$.

It is proved in [Nek08a, Theorem 6.6] that there exists k such that the map $I^{(k)}$ is homotopic (through equivariant maps) to a contracting map, hence the spaces $\overline{\Xi} \otimes \mathfrak{M}^{\otimes n}$ converge to the limit G -space \mathcal{X}_G , by Theorem 3.4.

We can replace $\overline{\Xi}$ by any sub-complex Ξ of $\overline{\Xi}$ (or of the barycentric subdivision of $\overline{\Xi}$) such that $\Xi \cdot g = \Xi$ for all $g \in G$, and $I(\Xi \otimes x) \subseteq \Xi$ for all $x \in X$. For instance, it is natural to consider the complex $\bigcap_{n \geq 1} I^{(n)}(\overline{\Xi} \otimes \mathfrak{M}^{\otimes n})$.

If the map I is contracting, then the complexes $J_n = \overline{\Xi} \otimes \mathfrak{M}^{\otimes n}/G$ approximate the limit space \mathcal{J}_G .

3.6. Iterated monodromy groups.

Definition 3.12. A *partial self-covering* is a covering map $f : \mathcal{M}_1 \rightarrow \mathcal{M}$, where \mathcal{M} is a topological space and \mathcal{M}_1 is a subset of \mathcal{M} .

More generally, a *topological automaton* is a covering of orbispaces $f : \mathcal{M}_1 \rightarrow \mathcal{M}$ together with a morphism $\iota : \mathcal{M}_1 \rightarrow \mathcal{M}$ (which is an embedding in the case of a partial self-covering). For details on the definition of coverings and morphisms of orbispaces, see [Nek05] and [Nek08a].

The iterated monodromy group of a partial self-covering is defined in the following way.

Definition 3.13. Let $f : \mathcal{M}_1 \rightarrow \mathcal{M}$ be a partial self-covering of a path-connected and locally path connected topological space \mathcal{M} . Let $t \in \mathcal{M}$ be a base-point. Denote by K_n the kernel of the monodromy action of $\pi_1(\mathcal{M}, t)$ on the fiber $f^{-n}(t)$

of the n th iteration of f . Then the iterated monodromy group $\text{IMG}(f)$ is the quotient of the group $\pi_1(\mathcal{M}, t)$ by the intersection $\bigcap_{n \geq 0} K_n$.

The iterated monodromy group acts naturally by automorphisms on the rooted tree of inverse images $\bigsqcup_{n \geq 0} f^{-n}(t)$ of t under the iterations of the partial self-covering f . The action can be computed using the following permutational $\pi_1(\mathcal{M}, t)$ -bimodule.

Define \mathfrak{M}_f as the set of homotopy classes in \mathcal{M} of the paths starting in t and ending in a preimage $z \in f^{-1}(t)$. Then the right action of $\pi_1(\mathcal{M}, t)$ on \mathfrak{M}_f is given by pre-pending the loops:

$$\ell \cdot \gamma = \ell\gamma,$$

for all $\ell \in \mathfrak{M}_f$ and $\gamma \in \pi_1(\mathcal{M}, t)$. We compose paths as maps: in a product $\ell\gamma$ the path γ is passed before ℓ . The left action is given by taking lifts of loops by f :

$$\gamma \cdot \ell = f^{-1}(\gamma)_\ell \ell,$$

where $f^{-1}(\gamma)_\ell$ is the lift of γ by f starting at the end of ℓ .

Let $X \subset \mathfrak{M}_f$ be a right orbit transversal, i.e., a collection of paths $\{\ell_z\}_{z \in f^{-1}(t)}$ starting at t and ending in each of the preimages of t . The transversal defines a wreath recursion ψ_f on $\pi_1(\mathcal{M}, t)$, as it is described above (just after Definition 3.3). This recursion is the main method of encoding the iterated monodromy group.

It is sufficient, by Propositions 3.1 and 3.2, to know the virtual endomorphism associated with the wreath recursion in order to be able to reconstruct the wreath recursion. In many cases this is a convenient way to compute the iterated monodromy group. One can use the following proposition (see [Nek05]).

Proposition 3.6. *Let $f : \mathcal{M}_1 \rightarrow \mathcal{M}$ be a partial self-covering and suppose that \mathcal{M}_1 and \mathcal{M} are path connected and locally path connected. Then the virtual endomorphism of $\pi_1(\mathcal{M})$ associated with the permutational bimodule \mathfrak{M}_f is equal to the composition $\iota_* \circ f_*^{-1}$, where f_*^{-1} is the virtual homomorphism $\pi_1(\mathcal{M}) \rightarrow \pi_1(\mathcal{M}_1)$ lifting loops by f , and $\iota : \mathcal{M}_1 \rightarrow \mathcal{M}$ is the identical embedding. All morphisms of the fundamental groups are defined here up to inner automorphisms.*

The associated self-similar action of $\pi_1(\mathcal{M}, t)$ on X^* is conjugate to the action of $\pi_1(\mathcal{M}, t)$ on the tree of preimages of t , hence the iterated monodromy group $\text{IMG}(f)$ coincides with the faithful quotient of $\pi_1(\mathcal{M}, t)$ with respect to the wreath recursion ψ_f .

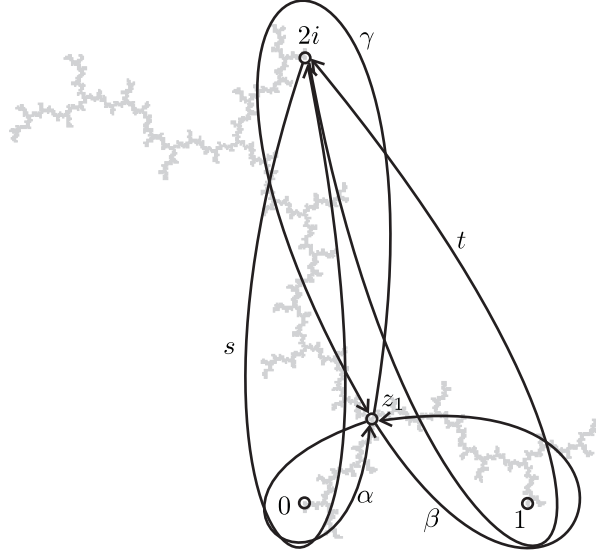
The main application of the iterated monodromy groups is based on the following theorem, proved in [Nek05] (which can also be deduced from Theorem 3.4 above).

Theorem 3.7. *Let $f : \mathcal{M}_1 \rightarrow \mathcal{M}$ be a partial self-covering of a path-connected and locally simply connected orbispace \mathcal{M} with a complete length metric. Suppose that the fundamental group of \mathcal{M} is finitely generated and f is uniformly expanding on \mathcal{M} .*

Then the iterated monodromy group $\text{IMG}(f)$ is contracting and the restriction of f onto the set of the accumulation points of $\bigcup_{n \geq 0} f^{-n}(t)$ is topologically conjugate with the limit dynamical system $s : \mathcal{J}_{\text{IMG}(f)} \rightarrow \mathcal{J}_{\text{IMG}(f)}$.

4. COMPUTATION OF THE ITERATED MONODROMY GROUP

Recall that the post-critical set V of f is the union of the line at infinity and the lines $z = 0$, $z = 1$, $w = 0$, $w = 1$, and $z = w$. It follows that the complement

FIGURE 2. Generators of $\pi_1(\mathbb{P}\mathbb{C}^2 \setminus V)$

$\mathbb{P}\mathbb{C}^2 \setminus V$ can be interpreted as the configuration space of a pair of points (z, w) in \mathbb{C} that are different from 0, 1, and from each other.

The rational map $(1 - \frac{z}{w})^2$ appearing in the second coordinate of f has three fixed points: $w = 1$, $w = 2i$, and $w = -2i$. The polynomial f_w for $w = 2i$ is conjugate to the polynomial $z^2 + i$.

The polynomial f_{2i} has two fixed points $z_1 \approx 0.3002 + 0.3752i$ and $z_2 \approx -1.3002 + 1.6248i$. Let us take $(z, w) = (z_1, 2i)$ as a base-point in the space $\mathbb{P}\mathbb{C}^2 \setminus V$.

Let α, β, γ be the loops in the configuration space $\mathbb{P}\mathbb{C}^2 \setminus V$ obtained by moving z around 0, 1, and $2i$, respectively; and let s and t be the loops obtained by moving w around 0 and 1, respectively. Then the fundamental group of $\mathbb{P}\mathbb{C}^2 \setminus V$ is generated by the loops α, β, γ, s , and t (see Figure 2).

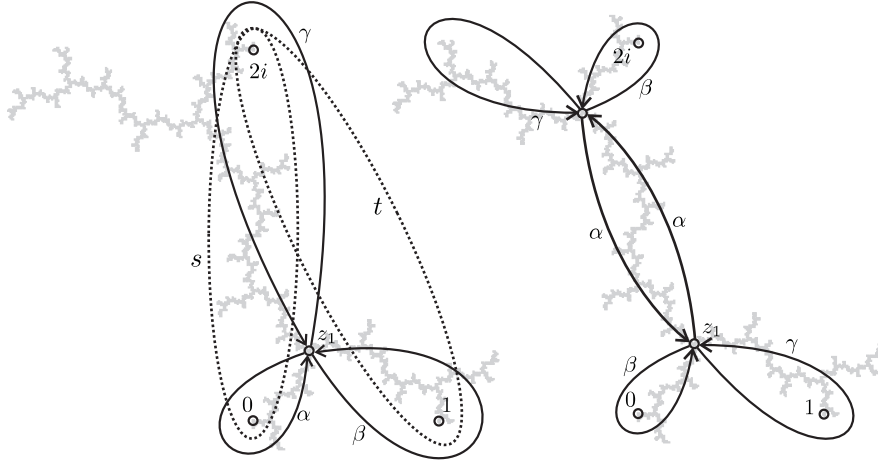
We have the following relations between these loops

$$\begin{aligned} (1) \quad & t\alpha t^{-1} = \alpha, & s\alpha s^{-1} &= \alpha\gamma\alpha\gamma^{-1}\alpha^{-1}, \\ (2) \quad & t\beta t^{-1} = \gamma\beta\gamma^{-1}, & s\beta s^{-1} &= \beta, \\ (3) \quad & t\gamma t^{-1} = \gamma\beta\gamma\beta^{-1}\gamma^{-1}, & s\gamma s^{-1} &= \alpha\gamma\alpha^{-1}, \end{aligned}$$

since s and t correspond to the Dehn twists around the curves shown on the left-hand side part of Figure 3.

Lemma 4.1. *The subgroup of $\pi_1(\mathbb{P}\mathbb{C}^2 \setminus V)$ generated by α, β , and γ is normal and has trivial centralizer.*

Proof. The group $G = \langle \alpha, \beta, \gamma \rangle$ is the fundamental group of the configuration space of one point z in $\mathbb{C} \setminus \{0, 1, w\}$, where $w \in \mathbb{C}$ is an arbitrary point different from 0 and 1. By [Bir74, Theorem 1.4], the subgroup $G < \pi_1(\mathbb{P}\mathbb{C}^2 \setminus V)$ is normal with the quotient isomorphic to the configuration space of one point p in $\mathbb{C} \setminus \{0, 1\}$. It follows that G is the fundamental group of a three-punctured plane, and the quotient $\pi_1(\mathbb{P}\mathbb{C}^2 \setminus V)/G$ is the fundamental group of a two-punctured plane. Consequently

FIGURE 3. Computation of $\text{IMG}(f)$

G and $\pi_1(\mathbb{P}\mathbb{C}^2 \setminus V)/G$ are free of rank 3 and 2, respectively. Hence the group $\langle s, t \rangle$ is *a fortiori* free.

It is known (see, for instance [Bir74, Corollary 1.8.3]) that the braid group B_n acts faithfully on the free group F_n by automorphisms in the natural way. In particular, the action of $\langle s, t \rangle$ on G by conjugation is faithful. It follows that if $g \in \pi_1(\mathbb{P}\mathbb{C}^2 \setminus V)$ acts trivially by conjugation on G , then it belongs to G . But G is free, hence only the trivial element of $\pi_1(\mathbb{P}\mathbb{C}^2 \setminus V)$ centralizes G . \square

In other words, if we know that two elements g_1 and g_2 of $\pi_1(\mathbb{P}\mathbb{C}^2 \setminus V)$ define the same automorphism on $\langle \alpha, \beta, \gamma \rangle$ by conjugation, then we know that $g_1 = g_2$. We will use this fact to identify the elements of the fundamental group by their action on the free group $\langle \alpha, \beta, \gamma \rangle$.

Proposition 4.2. *The values of the virtual endomorphism of $\pi_1(\mathbb{P}\mathbb{C}^2 \setminus V)$ associated with the partial self-covering f on the generators of its domain are*

$$\begin{aligned} \phi(\alpha^2) &= \varepsilon, & \phi(\beta) &= \alpha, & \phi(\gamma) &= \beta, & \phi(\alpha\beta\alpha^{-1}) &= \gamma, & \phi(\alpha\gamma\alpha^{-1}) &= \varepsilon \\ \phi(s^2) &= \varepsilon, & \phi(t) &= \beta\alpha\beta^{-1}\gamma\beta t^{-1}s^{-1}, & \phi(sts^{-1}) &= t. \end{aligned}$$

Recall that ε denotes the identity element of the group.

Proof. The domain of ϕ is the subgroup of loops such that their f -preimages starting in $(z_1, 2i)$ are again loops. It is a subgroup of index 4, since the covering is 4-fold.

The right-hand side of Figure 3 shows the preimages of the loops α, β, γ under the action of the polynomial f_{2i} (the labels show the images of the corresponding paths under the action of f_{2i}). We see that $\alpha^2, \beta, \gamma, \alpha\beta\alpha^{-1}$ and $\alpha\gamma\alpha^{-1}$ belong to the domain of ϕ . It is also clear that s^2, t and sts^{-1} belong to the domain of ϕ . These elements already generate a subgroup of index 4 in the fundamental group of $\mathbb{P}\mathbb{C}^2 \setminus V$, due to relations (1)–(3) between α, β, γ , and s, t . Consequently, these elements generate the domain of ϕ .

We see from Figure 3 that

$$\phi(\alpha^2) = \varepsilon, \quad \phi(\beta) = \alpha, \quad \phi(\gamma) = \beta, \quad \phi(\alpha\beta\alpha^{-1}) = \gamma, \quad \phi(\alpha\gamma\alpha^{-1}) = \varepsilon.$$

It is more convenient to find the action of ϕ on rest of the generators of the domain using the action of s and t on $\langle \alpha, \beta, \gamma \rangle$ (see Lemma 4.1).

We have

$$\phi(s^2)\alpha\phi(s^2)^{-1} = \phi(s^2\beta s^{-2}) = \phi(\beta) = \alpha,$$

$$\begin{aligned} \phi(s^2)\beta\phi(s^{-2}) &= \phi(s^2\gamma s^{-2}) = \phi(\alpha\gamma\alpha\gamma\alpha^{-1}\gamma^{-1}\alpha^{-1}) = \\ &= \phi(\alpha\gamma\alpha^{-1} \cdot \alpha^2 \cdot \gamma \cdot \alpha^{-2} \cdot \alpha\gamma^{-1}\alpha^{-1}) = \beta, \end{aligned}$$

and

$$\begin{aligned} \phi(s^2)\gamma\phi(s^{-2}) &= \phi(s^2\alpha\beta\alpha^{-1}s^2) = \\ &= \phi((\alpha\gamma)^2\alpha(\alpha\gamma)^{-2}\beta(\alpha\gamma)^2\alpha^{-1}(\alpha\gamma)^{-2}) = \\ &= \phi(\alpha\gamma\alpha^{-1} \cdot \alpha^2 \cdot \gamma \cdot \alpha\gamma^{-1}\alpha^{-1} \cdot \gamma^{-1} \cdot \alpha^{-1}\beta\alpha \cdot \gamma \cdot \alpha\gamma\alpha^{-1} \cdot \gamma^{-1} \cdot \alpha^{-2} \cdot \alpha\gamma^{-1}\alpha^{-1}) = \\ &= \beta\beta^{-1}\gamma\beta\beta^{-1} = \gamma, \end{aligned}$$

which implies that

$$\phi(s^2) = \varepsilon.$$

We have

$$\begin{aligned} \phi(t)\alpha\phi(t)^{-1} &= \phi(t\beta t^{-1}) = \phi(\gamma\beta\gamma^{-1}) = \beta\alpha\beta^{-1}, \\ \phi(t)\beta\phi(t)^{-1} &= \phi(t\gamma t^{-1}) = \phi(\gamma\beta\gamma\beta^{-1}\gamma^{-1}) = \beta\alpha\beta\alpha^{-1}\beta^{-1}, \end{aligned}$$

and

$$\phi(t)\gamma\phi(t)^{-1} = \phi(t\alpha\beta\alpha^{-1}t^{-1}) = \phi(\alpha\gamma\beta\gamma^{-1}\alpha^{-1}) = \gamma.$$

It follows that

$$\phi(t) = r = \beta\alpha\beta^{-1}\gamma\beta t^{-1}s^{-1},$$

since direct computations show that

$$r\alpha r^{-1} = \beta\alpha\beta^{-1}, \quad r\beta r^{-1} = \beta\alpha\beta\alpha^{-1}\beta^{-1}, \quad r\gamma r^{-1} = \gamma.$$

It remains to compute $\phi(sts^{-1})$. We have

$$\begin{aligned} \phi(sts^{-1})\alpha\phi(sts^{-1})^{-1} &= \phi(sts^{-1}\beta st^{-1}s^{-1}) = \phi(s\gamma\beta\gamma^{-1}s^{-1}) = \\ &= \phi(\alpha\gamma\alpha^{-1}\beta\alpha\gamma^{-1}\alpha^{-1}) = \alpha, \end{aligned}$$

$$\begin{aligned} \phi(sts^{-1})\beta\phi(sts^{-1})^{-1} &= \phi(sts^{-1}\gamma st^{-1}s^{-1}) = \\ &= \phi(st\gamma^{-1}\alpha^{-1}\gamma\alpha\gamma t^{-1}s^{-1}) = \\ &= \phi(s\gamma\beta\gamma^{-1}\beta^{-1}\gamma^{-1}\alpha^{-1}\gamma\beta\gamma\beta^{-1}\gamma^{-1}\alpha\gamma\beta\gamma\beta^{-1}\gamma^{-1}s) = \\ &= \phi(\alpha\gamma\alpha^{-1}\beta\alpha\gamma^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1}\beta\alpha\gamma\alpha^{-1}\beta^{-1}\alpha\beta\alpha\gamma\alpha^{-1}\beta^{-1}\alpha\gamma^{-1}\alpha^{-1}) = \gamma\beta\gamma^{-1}, \end{aligned}$$

and

$$\begin{aligned} \phi(sts^{-1})\gamma\phi(sts^{-1})^{-1} &= \phi(sts^{-1}\alpha\beta\alpha^{-1}st^{-1}s^{-1}) = \\ &= \phi(\alpha\gamma\alpha^{-1}\beta\alpha\gamma^{-1}\alpha^{-1}\beta^{-1}\alpha\beta\alpha\gamma\alpha^{-1}\beta\alpha\gamma^{-1}\alpha^{-1}\beta^{-1}\alpha^{-1}\beta\alpha\gamma\alpha^{-1}\beta^{-1}\alpha\gamma^{-1}\alpha^{-1}) = \\ &= \gamma\beta\gamma\beta^{-1}\gamma^{-1}, \end{aligned}$$

which implies that

$$\phi(sts^{-1}) = t,$$

which finishes the proof of the proposition. \square

Theorem 4.3. *The iterated monodromy group $\text{IMG}(f)$ is given by the wreath recursion*

$$\begin{aligned}\alpha &= \sigma, & \beta &= (\alpha, \gamma, \alpha, \beta^{-1}\gamma\beta), & \gamma &= (\beta, \varepsilon, \varepsilon, \beta), \\ t &= (r, r, t, t), & s &= \pi(\varepsilon, \beta^{-1}, \varepsilon, \beta),\end{aligned}$$

where $\sigma = (12)(34)$, $\pi = (14)(23)$, and

$$r = \beta\alpha\beta^{-1}\gamma\beta t^{-1}s^{-1}.$$

Proof. We will use Proposition 3.2 to find the wreath recursion with the associated virtual endomorphism given in Proposition 4.2.

The set $\{\varepsilon, \alpha, s, \alpha s\}$ is a left coset transversal of $\pi_1(\mathbb{P}\mathbb{C}^2 \setminus V)$ modulo the domain of the virtual endomorphism ϕ . Let us take the transversal in the given order. Using Propositions 3.2 and 4.2 and relations (1)–(3), we get:

$$\begin{aligned}\alpha &= \sigma(\phi(\alpha^{-1}\alpha), \phi(\alpha^2), \phi((\alpha s)^{-1}(\alpha s)), \phi(s^{-1}\alpha^2 s)) = \\ &\quad \sigma(\phi(\varepsilon), \phi(\alpha^2), \phi(\varepsilon), \phi(\gamma^{-1}\alpha^2\gamma)) = \sigma, \\ \beta &= (\phi(\beta), \phi(\alpha^{-1}\beta\alpha), \phi(s^{-1}\beta s), \phi(s^{-1}\alpha^{-1}\beta\alpha s)) = \\ &\quad (\phi(\beta), \phi(\alpha^{-1}\beta\alpha), \phi(\beta), \phi(\gamma^{-1}\alpha^{-1}\gamma\beta\gamma^{-1}\alpha\gamma)) = (\alpha, \gamma, \alpha, \beta^{-1}\gamma\beta), \\ \gamma &= (\phi(\gamma), \phi(\alpha^{-1}\gamma\alpha), \phi(s^{-1}\gamma s), \phi(s^{-1}\alpha^{-1}\gamma\alpha s)) = \\ &\quad (\phi(\gamma), \phi(\alpha^{-1}\gamma\alpha), \phi(\gamma^{-1}\alpha^{-1}\gamma\alpha\gamma), \phi(\gamma^{-1}\alpha^{-2}\gamma\alpha^2\gamma)) = (\beta, \varepsilon, \varepsilon, \beta), \\ t &= (\phi(t), \phi(\alpha^{-1}t\alpha), \phi(s^{-1}ts), \phi(s^{-1}\alpha^{-1}t\alpha s)) = \\ &\quad (\phi(t), \phi(t), \phi(s^{-1}ts), \phi(s^{-1}ts)) = (r, r, t, t), \\ s &= \pi(\phi(s^{-1}s), \phi(s^{-1}\alpha^{-1}s\alpha), \phi(s^2), \phi(\alpha^{-1}s\alpha s)) = \\ &\quad \pi(\phi(1), \phi(\gamma^{-1}\alpha^{-1}\gamma\alpha), \phi(s^2), \phi(\gamma\alpha\gamma^{-1}\alpha^{-1}s^2)) = \pi(\varepsilon, \beta^{-1}, \varepsilon, \beta).\end{aligned}$$

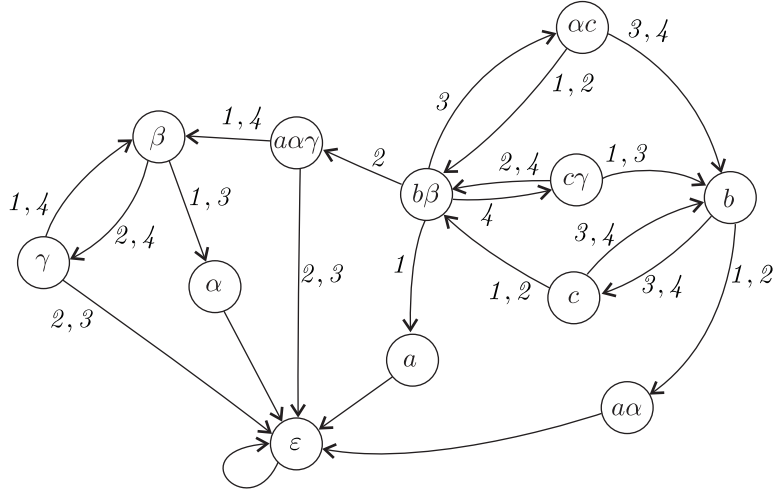
□

5. POLYHEDRAL MODEL OF f

5.1. Index two extension. The function f has real coefficients, hence complex conjugation of both coordinates is an automorphism of the dynamical system $(f, \mathbb{P}\mathbb{C}^2)$. We can take the quotient of this dynamical system by this automorphism (i.e., by the group of order two generated by it). The iterated monodromy group of the quotient is, by general theory (see [Nek05, Theorem 3.7.1] and [Nek08b, Subsection 3.8]), an index two extension of $\text{IMG}(f)$.

This extension was considered in [Nek07] and was used to study the properties of the Cantor set of groups associated with the iterations of the polynomials f_{p_n} . It is the group Γ generated by the transformations

$$\begin{aligned}(4) \quad & \alpha = \sigma, & a &= \pi, \\ (5) \quad & \beta = (\alpha, \gamma, \alpha, \gamma), & b &= (a\alpha, a\alpha, c, c), \\ (6) \quad & \gamma = (\beta, \varepsilon, \varepsilon, \beta), & c &= (b\beta, b\beta, b, b),\end{aligned}$$

FIGURE 4. Automaton \mathfrak{A} generating Γ

where $\sigma = (12)(34)$ and $\pi = (13)(24)$, as before. We will not need the fact that Γ is really the iterated monodromy group of the quotient of f by complex conjugation, so we will not present its proof here.

Note that:

$$(7) \quad b\beta = (a, a\alpha\gamma, \alpha c, c\gamma), \quad a\alpha = \pi\sigma, \quad c\gamma = (b, b\beta, b, b\beta),$$

$$(8) \quad a\alpha\gamma = \pi\sigma(\beta, \varepsilon, \varepsilon, \beta), \quad \alpha c = \sigma(b\beta, b\beta, b, b).$$

We see that the set $\{\alpha, \beta, \gamma, a, b, c, a\alpha, b\beta, c\gamma, a\alpha\gamma, \alpha c, \varepsilon\}$ is state-closed, hence it is the set of states of an automaton \mathfrak{A} generating the group Γ . The Moore diagram of the automaton \mathfrak{A} is shown on Figure 4. The labels on the arrows show the input letters. An arrow with two labels i, j correspond to two arrows with labels i and j . Arrows without labels correspond to four arrows with labels $1, 2, 3$, and 4 . The action of the states on the first level is not shown on the figure (but it follows from their labels).

Direct computation shows that all generators are involutions and that the following relations hold

$$\begin{aligned} \alpha^a &= \alpha, & \alpha^b &= \alpha, & \alpha^c &= \alpha, \\ \beta^a &= \beta, & \beta^b &= \beta, & \beta^c &= \beta\gamma, \\ \gamma^a &= \gamma^\alpha, & \gamma^b &= \gamma^\beta, & \gamma^c &= \gamma. \end{aligned}$$

Let us show that the iterated monodromy group of f is a self-similar subgroup of index 2 in Γ . It follows from the relations mentioned above that the group generated by $\alpha, \beta, \gamma, ac, bc$ is a subgroup of index two. Let us show that it is also self-similar (i.e., becomes state-closed after composition of the wreath recursion with an inner automorphism of the wreath product $\mathfrak{S}(X) \wr \Gamma$). We have

$$ac = \pi(b\beta, b\beta, b, b), \quad cb = (b\beta a\alpha, b\beta a\alpha, bc, bc).$$

Let us conjugate the right-hand side by $(\varepsilon, \varepsilon, b, b)$ (i.e., let us change the basis of the permutational bimodule from $\{1, 2, 3, 4\}$ to $\{1, 2, 3 \cdot b, 4 \cdot b\}$). We get

$$\begin{aligned}\alpha &= \sigma \\ \beta &= (\alpha, \gamma, \alpha, \gamma^b) = (\alpha, \gamma, \alpha, \gamma^\beta) \\ \gamma &= (\beta, \varepsilon, \varepsilon, \beta^b) = (\beta, \varepsilon, \varepsilon, \beta),\end{aligned}$$

and

$$\begin{aligned}ac &= \pi(\beta, \beta, \varepsilon, \varepsilon) \\ cb &= (\beta\alpha b a, \beta\alpha b a, cb, cb).\end{aligned}$$

If we denote $s = ac\gamma$ and $t = cb$, then we have

$$s = ac\gamma = \pi(\varepsilon, \beta, \varepsilon, \beta)$$

and $\beta\alpha b a = \beta\alpha b c\gamma\gamma c a = \beta\alpha\beta\gamma\beta b c\gamma c a = \beta\alpha\beta\gamma\beta t^{-1}s^{-1}$, so that

$$t = (r, r, t, t),$$

where $r = \beta\alpha b a = \beta\alpha\beta\gamma\beta t^{-1}s^{-1}$, as in Theorem 4.3.

We have just proved that the index two subgroup $\langle \alpha, \beta, \gamma, ac, cb \rangle$ is isomorphic as a self-similar group to $\text{IMG}(f)$.

5.2. Some finite subgroups of Γ . Direct computations (see also [Nek07]) show that the following relations hold in the group $\Gamma = \langle \alpha, \beta, \gamma, a, b, c \rangle$.

$$(\alpha\gamma)^4 = \varepsilon, \quad (\alpha\beta)^8 = \varepsilon, \quad (\beta\gamma)^8 = \varepsilon$$

and

$$\begin{aligned}(ac)^2 &= (\beta, \beta, \beta, \beta) = (\alpha\gamma)^2 \\ (ab)^4 &= (ca\alpha, ca\alpha, a\alpha c, a\alpha c)^2 = ((\alpha\gamma)^2, (\alpha\gamma)^2, (\alpha\gamma)^2, (\alpha\gamma)^2) = (\alpha\beta)^4 \\ (bc)^4 &= ((ab)^4(\alpha\beta)^4, (ab)^4(\alpha\beta)^4, (cb)^4, (cb)^4) = ((\alpha\beta)^8, (\alpha\beta)^8, (cb)^4, (cb)^4) = \varepsilon.\end{aligned}$$

Consequently, the products ab , ac , and bc are of orders 8, 4, and 4, respectively. Note that the elements $a\alpha$, $b\beta$, $c\gamma$ are of order 2 (since $[a, \alpha] = [b, \beta] = [c, \gamma] = \varepsilon$) and that we have

$$\begin{aligned}(ac\gamma)^2 &= ac\gamma ac\gamma = aca\alpha\gamma\alpha c\gamma = acac\alpha\gamma\alpha\gamma = (\alpha\gamma)^4 = \varepsilon, \\ (a\alpha b\beta)^2 &= (ab)^2(\alpha\beta)^2, \quad (a\alpha b\beta)^4 = (ab)^4(\alpha\beta)^4 = (\alpha\beta)^8 = \varepsilon\end{aligned}$$

hence $a \cdot c\gamma$ is of order 2, while $a\alpha \cdot b\beta$ is of order 4.

It follows that the group Γ contains the following finite groups

$$\begin{aligned}\Gamma_{A_1} &= \langle \alpha, b, c \rangle = \langle \alpha \rangle \times \langle b, c \rangle \cong C_2 \times D_4, \\ \Gamma_{B_1} &= \langle \beta, a, c\gamma \rangle = \langle \beta \rangle \times \langle a, c\gamma \rangle \cong C_2 \times D_2, \\ \Gamma_{C_1} &= \langle \gamma, a\alpha, b\beta \rangle = \langle \gamma \rangle \times \langle a\alpha, b\beta \rangle \cong C_2 \times D_4,\end{aligned}$$

and

$$\begin{aligned}\Gamma_A &= \langle \beta, \gamma, b, c \rangle = \langle \beta, \gamma \rangle \rtimes \langle b, c \rangle \cong D_8 \rtimes D_4, \\ \Gamma_B &= \langle \alpha, \gamma, a, c \rangle = \langle \alpha, \gamma \rangle \rtimes \langle a, c\gamma \rangle \cong D_4 \rtimes D_2, \\ \Gamma_C &= \langle \alpha, \beta, a, b \rangle = \langle \alpha, \beta \rangle \rtimes \langle a\alpha, b\beta \rangle \cong D_8 \rtimes D_4,\end{aligned}$$

where in the last three cases in a semidirect product $\langle x_1, y_1 \rangle \rtimes \langle x_2, y_2 \rangle$ the generators x_2 and y_2 act on $\langle x_1, y_1 \rangle$ as conjugation by x_1 and y_1 , respectively. (The group of inner automorphisms of D_{2n} is isomorphic to D_n .)

5.3. Nucleus of Γ . It was proved in [Nek07] that the group Γ is contracting without presenting the nucleus explicitly. More careful analysis gives the following complete description of the nucleus of Γ .

Proposition 5.1. *The set $\mathcal{N} = \Gamma_A \cup \Gamma_B \cup \Gamma_C \cup \Gamma_{A_1} \cup \Gamma_{B_1} \cup \Gamma_{C_1}$ is the nucleus of the group Γ .*

Proof. It is checked directly that the set \mathcal{N} is state-closed. It is sufficient to show that sections of $\mathcal{N} \cdot \{\alpha, \beta, \gamma, a, b, c\}$ eventually belong to \mathcal{N} . Since sections of α and a are trivial, it is sufficient to consider the set $\mathcal{N} \cdot \{\beta, \gamma, b, c\}$.

The generators of Γ_A are β, γ, b, c , hence $\Gamma_A \cdot \{\beta, \gamma, b, c\} = \Gamma_A$. The group Γ_B is generated by $\alpha, \gamma = (\beta, \varepsilon, \varepsilon, \beta), a, c = (b\beta, b\beta, b, b)$. It follows that the first level sections of the elements of Γ_B belong to $\langle b, \beta \rangle$. Consequently, the first level sections of $\Gamma_B \cdot \{\beta, \gamma, b, c\} = \Gamma_B \cdot \{\beta, b\}$ belong either to $\langle b, \beta, \alpha \rangle < \Gamma_C$, or to $\langle b, \beta, \gamma \rangle < \Gamma_A$, or to $\langle b, \beta, a\alpha \rangle < \Gamma_C$, or to $\langle b, \beta, c \rangle < \Gamma_A$.

Note that $b\beta = (a, a\alpha\gamma, c\alpha, c\gamma)$, therefore the sections of $\Gamma_B \cdot b\beta$ belong to

$$\langle b, \beta, a \rangle \cup \langle b, \beta, c\gamma \rangle \cup \langle b, \beta, c\alpha \rangle \cup \langle b, \beta\gamma a\alpha \rangle \subset \Gamma_C \cup \Gamma_A \cup \Gamma_A \cdot \alpha \cup \Gamma_A \cdot a\alpha.$$

Consequently, the second level sections of $\Gamma_B \cdot b\beta$ belong to \mathcal{N} .

The group Γ_C is generated by $\alpha, \beta = (\alpha, \gamma, \alpha, \gamma), a, b = (a\alpha, a\alpha, c, c)$. The first level sections of the elements of Γ_C belong to $\langle a, \alpha, \gamma, c \rangle = \Gamma_B$. The first level sections of $\Gamma_C \cdot \{\beta, \gamma, b, c\} = \Gamma_C \cdot \{\gamma, c\}$ belong to $\Gamma_B \cdot \{\beta, b, b\beta\}$, hence the third level sections of Γ_C belong to \mathcal{N} .

The group Γ_{A_1} is generated by $\alpha, b = (a\alpha, a\alpha, c, c), c = (b\beta, b\beta, b, b)$, hence the first level sections of $\Gamma_{A_1} \cdot \{\beta, \gamma, b, c\} = \Gamma_{A_1} \cdot \{\beta, \gamma\}$ belong to

$$(\langle a\alpha, b\beta \rangle \cup \langle b, c \rangle) \cdot \{\alpha, \beta, \gamma\} \subset \langle \alpha, \beta, a, b \rangle \cup \langle \gamma, a\alpha, b\beta \rangle \cup \langle \alpha, b, c \rangle \cup \langle \beta, \gamma, b, c \rangle \subset \mathcal{N}.$$

The group Γ_{B_1} is generated by $\beta = (\alpha, \gamma, \alpha, \gamma), a, c\gamma = (b, b\beta, b, b\beta)$. The first level sections of Γ_{B_1} belong to $\langle \alpha, b \rangle \cup \langle \gamma, b\beta \rangle$. Hence, the first level sections of $\Gamma_{B_1} \cdot \{\beta, \gamma, b, c\} = \Gamma_{B_1} \cdot \{\gamma, b, c\}$ belong to

$$(\langle \alpha, b \rangle \cup \langle \gamma, b\beta \rangle) \cdot \{\beta, a\alpha, b\beta, b, c\} \subset \langle \alpha, \beta, a, b \rangle \cup \langle \alpha, b, c \rangle \cup \langle \beta, \gamma, b, c \rangle \cup \langle \gamma, a\alpha, b\beta \rangle,$$

which is a subset of \mathcal{N} .

The group Γ_{C_1} is generated by $\gamma = (\beta, \varepsilon, \varepsilon, \beta), a\alpha, b\beta = (a, a\alpha\gamma, c\alpha, c\gamma)$, which implies that the first level sections of Γ_{C_1} belong to $\langle \beta, a, c\gamma \rangle \cup \langle a\alpha\gamma, c\alpha \rangle$. Therefore, the first level sections of the elements of $\Gamma_{C_1} \cdot \{\beta, \gamma, b, c\} = \Gamma_{C_1} \cdot \{\beta, b, c\}$ belong to

$$(\langle \beta, a, c\gamma \rangle \cup \langle a\alpha\gamma, c\alpha \rangle) \cdot \{\alpha, \gamma, a\alpha, b\beta, b\}.$$

Since the sections of a and α are trivial, the sections of the elements of $\langle \beta, a, c\gamma \rangle \cdot \{\alpha, \gamma, a\alpha, b\beta, b\}$ in non-empty words are the same as sections of

$$\langle \beta, a, c\gamma \rangle \cdot \{\gamma, b\beta, b\} = \langle \beta, a, c\gamma \rangle \cdot \{\gamma, b\} = \Gamma_{B_1} \cdot \{\gamma, b\},$$

but we have seen that sections of the elements of this set in one-letter words belong to \mathcal{N} .

It remains to consider the set $\langle a\alpha\gamma, c\alpha \rangle \cdot \{\alpha, \gamma, a\alpha, b\beta, b\}$. We have

$$\langle a\alpha\gamma, c\alpha \rangle \cdot \{\alpha, \gamma, a\alpha, b\beta, b\} \subset \langle \alpha, \gamma, a, c \rangle \cdot \{\varepsilon, b\beta, b\} = \Gamma_B \cdot \{\varepsilon, b\beta, \beta\},$$

but we have seen above that the sections of the elements of the set $\Gamma_B \cdot \{\varepsilon, b\beta, \beta\}$ in words of length two belong to \mathcal{N} . \square

The sizes of the defined subgroups of \mathcal{N} are $|\Gamma_A| = 128$, $|\Gamma_B| = 32$, $|\Gamma_C| = 128$, $|\Gamma_{A_1}| = 16$, $|\Gamma_{B_1}| = 8$, $|\Gamma_{C_1}| = 16$. Their pairwise intersections are (here we denote by Γ_{XY} the subgroup $\Gamma_X \cap \Gamma_Y$):

$$\begin{aligned}\Gamma_{AB} &= \langle \gamma, c \rangle, & \Gamma_{AC} &= \langle \beta, b \rangle, & \Gamma_{BC} &= \langle \alpha, a \rangle, \\ \Gamma_{AA_1} &= \langle b, c \rangle, & \Gamma_{AB_1} &= \langle \beta, c\gamma \rangle, & \Gamma_{AC_1} &= \langle \gamma, b\beta \rangle, \\ \Gamma_{BA_1} &= \langle \alpha, c \rangle, & \Gamma_{BB_1} &= \langle a, c\gamma \rangle, & \Gamma_{BC_1} &= \langle \gamma, a\alpha \rangle, \\ \Gamma_{CA_1} &= \langle \alpha, b \rangle, & \Gamma_{CB_1} &= \langle \beta, a \rangle, & \Gamma_{CC_1} &= \langle a\alpha, b\beta \rangle.\end{aligned}$$

The intersections $\Gamma_{A_1} \cap \Gamma_{B_1}$, $\Gamma_{A_1} \cap \Gamma_{C_1}$, $\Gamma_{B_1} \cap \Gamma_{C_1}$ are trivial. All non-trivial pairwise intersections are isomorphic to $C_2 \times C_2$, except for

$$\Gamma_{AA_1} \cong D_4, \quad \Gamma_{BB_1} \cong D_2, \quad \Gamma_{CC_1} \cong D_4.$$

The only non-trivial triple intersections (all of order 2) are

$$\Gamma_{A_1BC} = \langle \alpha \rangle, \quad \Gamma_{AB_1C} = \langle \beta \rangle, \quad \Gamma_{ABC_1} = \langle \gamma \rangle$$

and

$$\begin{aligned}\Gamma_{A_1AB} &= \langle c \rangle, & \Gamma_{A_1AC} &= \langle b \rangle, & \Gamma_{B_1AB} &= \langle c\gamma \rangle, \\ \Gamma_{B_1BC} &= \langle a \rangle, & \Gamma_{C_1BC} &= \langle a\alpha \rangle, & \Gamma_{C_1AC} &= \langle b\beta \rangle.\end{aligned}$$

Since all triple intersections are of order 2 and are pairwise different, there are no non-trivial intersections of four or more different groups Γ_X .

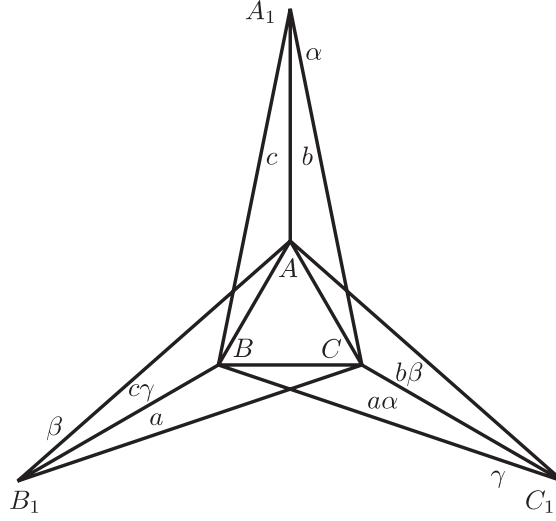
Removing the identity and using the inclusion-exclusion formula, we get that the size of the nucleus of Γ is

$$1 + 127 + 31 + 127 + 15 + 7 + 15 - 7 - 3 - 7 - 9 \cdot 3 + 9 = 288.$$

5.4. Sections of subgroups of \mathcal{N} . Let us list sections $G|_x$ for the groups Γ_* .

	G	$G _1$	$G _2$	$G _3$	$G _4$
Γ_A	$\langle \beta, \gamma, b, c \rangle$	$\langle \alpha, \beta, a, b \rangle$	$\langle \gamma, a\alpha, b\beta \rangle$	$\langle \alpha, b, c \rangle$	$\langle \beta, \gamma, b, c \rangle$
Γ_B	$\langle \alpha, \gamma, a, c \rangle$	$\langle \beta, b \rangle$	$\langle \beta, b \rangle$	$\langle \beta, b \rangle$	$\langle \beta, b \rangle$
Γ_C	$\langle \alpha, \beta, a, b \rangle$	$\langle \alpha, \gamma, a, c \rangle$	$\langle \alpha, \gamma, a, c \rangle$	$\langle \alpha, \gamma, a, c \rangle$	$\langle \alpha, \gamma, a, c \rangle$
Γ_{A_1}	$\langle \alpha, b, c \rangle$	$\langle a\alpha, b\beta \rangle$	$\langle a\alpha, b\beta \rangle$	$\langle b, c \rangle$	$\langle b, c \rangle$
Γ_{B_1}	$\langle \beta, a, c\gamma \rangle$	$\langle \alpha, b \rangle$	$\langle \gamma, b\beta \rangle$	$\langle \alpha, b \rangle$	$\langle \gamma, b\beta \rangle$
Γ_{C_1}	$\langle \gamma, a\alpha, b\beta \rangle$	$\langle \beta, a, c\gamma \rangle$	$\langle a\alpha\gamma, \alpha c \rangle$	$\langle a\alpha\gamma, \alpha c \rangle$	$\langle \beta, a, c\gamma \rangle$

	G	$G _1$	$G _2$	$G _3$	$G _4$
Γ_{AB}	$\langle \gamma, c \rangle$	$\langle \beta, b \rangle$	$\langle b\beta \rangle$	$\langle b \rangle$	$\langle \beta, b \rangle$
Γ_{AC}	$\langle \beta, b \rangle$	$\langle \alpha, a \rangle$	$\langle \gamma, a\alpha \rangle$	$\langle \alpha, c \rangle$	$\langle \gamma, c \rangle$
Γ_{BC}	$\langle \alpha, a \rangle$	$\{\varepsilon\}$	$\{\varepsilon\}$	$\{\varepsilon\}$	$\{\varepsilon\}$
Γ_{AA_1}	$\langle b, c \rangle$	$\langle a\alpha, b\beta \rangle$	$\langle a\alpha, b\beta \rangle$	$\langle b, c \rangle$	$\langle b, c \rangle$
Γ_{AB_1}	$\langle \beta, c\gamma \rangle$	$\langle \alpha, b \rangle$	$\langle \gamma, b\beta \rangle$	$\langle \alpha, b \rangle$	$\langle \gamma, b\beta \rangle$
Γ_{AC_1}	$\langle \gamma, b\beta \rangle$	$\langle \beta, a \rangle$	$\langle a\alpha\gamma \rangle$	$\langle \alpha c \rangle$	$\langle \beta, c\gamma \rangle$
Γ_{BA_1}	$\langle \alpha, c \rangle$	$\langle b\beta \rangle$	$\langle b\beta \rangle$	$\langle b \rangle$	$\langle b \rangle$
Γ_{BB_1}	$\langle a, c\gamma \rangle$	$\langle b \rangle$	$\langle b\beta \rangle$	$\langle b \rangle$	$\langle b\beta \rangle$
Γ_{BC_1}	$\langle \gamma, a\alpha \rangle$	$\langle \beta \rangle$	$\{\varepsilon\}$	$\{\varepsilon\}$	$\langle \beta \rangle$
Γ_{CA_1}	$\langle \alpha, b \rangle$	$\langle \alpha, \gamma \rangle$	$\langle \alpha, \gamma \rangle$	$\langle \alpha, \gamma \rangle$	$\langle \alpha, \gamma \rangle$
Γ_{CB_1}	$\langle \beta, a \rangle$	$\langle a\alpha, c \rangle$	$\langle a\alpha, c \rangle$	$\langle a\alpha, c \rangle$	$\langle a\alpha, c \rangle$
Γ_{CC_1}	$\langle a\alpha, b\beta \rangle$	$\langle a, c\gamma \rangle$	$\langle a\alpha\gamma, \alpha c \rangle$	$\langle a\alpha\gamma, \alpha c \rangle$	$\langle a, c\gamma \rangle$

FIGURE 5. Complex \mathcal{T}_0 associated with \mathcal{N}

The triple intersections Γ_{XYZ} are generated by single elements $g \in \mathfrak{A}$, therefore their sections are described on the Moore diagram of the automaton \mathfrak{A} (see Figure 4).

5.5. Complex associated with the nucleus of Γ . Denote by \mathcal{G} the set of subgroups $\Gamma_A, \Gamma_B, \Gamma_C, \Gamma_{A_1}, \Gamma_{B_1}, \Gamma_{C_1}$, and all their pairwise and triple intersections.

Denote by Ξ the simplicial complex associated with the poset of the subsets of Γ of the form $G \cdot g$ for $G \in \mathcal{G}$ and $g \in \Gamma$. Then Ξ is a Γ -invariant sub-complex of the barycentric subdivision of the Cayley-Rips complex Ξ of Γ defined by the generating set \mathcal{N} (see Subsection 3.5).

Denote by \mathcal{T}_0 the simplicial complex of the poset \mathcal{G} . It is the subcomplex of Ξ spanned by the vertices corresponding to cosets containing the identity, i.e., it is the union of the simplices of Ξ containing the vertex corresponding to the trivial subgroup $\{\varepsilon\} \in \mathcal{G}$ of Γ .

It follows from the description of the set of groups \mathcal{G} that we can represent \mathcal{T}_0 as a union of three tetrahedra A_1ABC, B_1ABC, C_1ABC with a common face ABC , as it is shown on Figure 5. Every element Γ_X, Γ_{XY} , or Γ_{XYZ} of \mathcal{G} corresponds to (the barycenter of) the corresponding vertex X , edge XY , or triangle XYZ , respectively. We have labeled on Figure 5 the triangles of \mathcal{T}_0 by the generators of the respective groups Γ_{XYZ} .

Denote

$$\Xi_n = \Xi \otimes \mathfrak{M}^{\otimes n}, \quad \mathcal{T}_n = \mathcal{T}_0 \otimes \mathbf{X}^n \subset \Xi_n.$$

The set \mathcal{T}_n is a fundamental domain of the action of Γ on Ξ_n (since $\mathbf{X}^n = \mathbf{X}^{\otimes n}$ is a right orbit transversal of the action of Γ on $\mathfrak{M}^{\otimes n}$). The space \mathcal{T}_n is the quotient of the space $\mathcal{T}_0 \times \mathbf{X}^n$ by the identifications

$$(\xi, v) \sim (\xi \cdot g^{-1}, g(v))$$

for all $g \in \mathcal{N}$, $\xi \in \mathcal{T}_0$, and $v \in \mathbf{X}^n$ such that $g|_v = \varepsilon$ and $\xi \cdot g^{-1} \in \mathcal{T}_0$.

We will denote the points of \mathcal{T}_n in the same way as the points of Ξ_n : the point $\xi \otimes v$ is the equivalence class of the point $(\xi, v) \in \mathcal{T}_0 \times \mathbb{X}^n$.

For $g \in \Gamma$, denote $K_{g,n} = \mathcal{T}_n \cap \mathcal{T}_n \cdot g$, and let $\kappa_{g,n} : K_{g,n} \rightarrow K_{g^{-1},n}$ be restriction onto $K_{g,n}$ of the map $\xi \mapsto \xi \cdot g^{-1}$.

If $\mathcal{T}_0 \cap \mathcal{T}_0 \cdot g$ is non-empty for $g \in \Gamma$, then there exists $G \in \mathcal{G}$ such that $G \cdot g \in \mathcal{G}$. But then $g \in G \subset \mathcal{N}$, and $G \cdot g = G$. It follows that $K_{g,0}$ is non-empty only for $g \in \mathcal{N}$; $K_{g,0} = K_{g^{-1},0}$; and $\kappa_{g,0}$ is an identity map.

We have

$$\begin{aligned} K_{\alpha,0} &= A_1BC, & K_{\beta,0} &= B_1AC, & K_{\gamma,0} &= C_1AB, \\ K_{a,0} &= B_1BC, & K_{b,0} &= A_1AC, & K_{c,0} &= A_1AB, \\ K_{a\alpha,0} &= C_1CB, & K_{b\beta,0} &= C_1CA, & K_{c\gamma,0} &= B_1BA. \end{aligned}$$

The remaining sets $K_{g,0}$ for $g \in \mathfrak{A}$ are one-dimensional:

$$K_{\alpha c,0} = A_1B, \quad K_{a\alpha\gamma,0} = C_1B.$$

Proposition 5.2. *The set $K_{g,n}$ is non-empty only for $g \in \mathcal{N}$. The map $\kappa_{g,n} : K_{g,n} \rightarrow K_{g^{-1},n}$ is given by the condition*

$$\kappa_{g,n}(\xi \otimes v) = \xi \otimes h(v),$$

where $h \in \mathcal{N}$ is such that $h|_v = g$, and $\xi \in K_{h,0}$.

Proof. If $\kappa_{g,n}$ is defined on a point $\xi \otimes v$, for $\xi \in \mathcal{T}_0$ and $v \in \mathbb{X}^n$, then $\xi \otimes v \cdot g^{-1} = \xi_1 \otimes v_1$ for some $\xi_1 \in \mathcal{T}_0$ and $v_1 \in \mathbb{X}^n$. Then, by the definition of a tensor product, there exists $h \in \Gamma$ such that $\xi = \xi_1 \cdot h$ and $h \cdot v \cdot g^{-1} = v_1$ in $\mathfrak{M}^{\otimes n}$. The first equality implies, by the argument above, that $h \in \mathcal{N}$ and $\xi = \xi_1 \in K_{h,0}$. Then $v_1 = h(v)$ and $h|_v \cdot g^{-1} = \varepsilon$, which implies that $g = h|_v \in \mathcal{N}$. \square

The next technical lemma will be used several times in our paper.

Lemma 5.3. *For every point $\xi \in \mathcal{T}_n$ the stabilizer Γ_ξ of ξ in Γ belongs either to \mathcal{G} or to the set*

$$\mathcal{G}_1 = \{\langle a\alpha\gamma, \alpha c \rangle, \quad \langle a\alpha\gamma \rangle, \quad \langle \alpha c \rangle\}.$$

In particular, it is generated by a subset of \mathfrak{A} .

If $\xi \in \mathcal{T}_n$ belongs to $K_{g,n}$ for $g \in \mathcal{N} \setminus \mathfrak{A}$, then $\kappa_{g,n}(\xi) = \xi$, i.e., g belongs to Γ_ξ .

Proof. We have seen above that stabilizer of every point $\xi \in \mathcal{T}_0$ belongs to \mathcal{G} .

Let us prove our lemma by induction. Suppose that it is true for n , and let g be an element of the stabilizer of a point $\xi \otimes vx$ for $\xi \in \mathcal{T}_0$, $v \in \mathbb{X}^n$, and $x \in \mathbb{X}$. Then $\xi \otimes vx \cdot g = \xi \otimes vx$, which means that there exists $h \in \Gamma$ such that $\xi \otimes v = \xi \otimes v \cdot h$ and $h \cdot x = x \cdot g$. Consequently, elements of the stabilizer of $\xi \otimes vx$ are sections at x of the intersection of $\Gamma_{\xi \otimes v}$ with the stabilizer of x . In other words

$$\Gamma_{\xi \otimes vx} = \phi_x(\Gamma_{\xi \otimes v}),$$

where ϕ_x is the virtual endomorphism associated with Γ and $x \in \mathbb{X}$.

The generators of $G \in \mathcal{G}$ (as they are listed in Subsection 5.3) acting non-trivially on the first level belong to $\{\alpha, a, a\alpha\}$. Consequently, for every $g \in G$ and $x \in \mathbb{X}$ the section $g|_x$ is equal to $\phi_x(h)$ for some $h \in G$. Therefore, $\phi_x(G) = G|_x$ for all $G \in \mathcal{G}$. The groups $G|_x$ are listed in Subsection 5.4 (one has also to add the sections $\langle g \rangle|_x = \langle g|_x \rangle$ for $g \in \mathfrak{A}$).

We see that $G|_x \in \mathcal{G} \cup \mathcal{G}_1$ for all $G \in \mathcal{G}$. Sections $g|_x$ of the elements of the groups from \mathcal{G}_1 belong to $\langle \beta, b \rangle$. But all subgroups of $\langle \beta, b \rangle$ belong to \mathcal{G} . It follows that stabilizers of points of \mathcal{T}_n belong to $\mathcal{G} \cup \mathcal{G}_1$ for all n .

Let us prove the remaining part of the lemma. Let $\xi \in \mathcal{T}_0$, $v \in \mathbf{X}^n$, and $x \in \mathbf{X}$ are such that $\xi \otimes vx$ belongs to $K_{g,n+1}$ for $g \in \mathcal{N} \setminus \mathfrak{A}$. Then there exists $h \in \mathcal{N}$ such that $\xi \cdot h = \xi$ and $h|_{vx} = g$. Since \mathfrak{A} is state-closed, $h|_v \notin \mathfrak{A}$. We have then $\xi \otimes v = \xi \otimes h \cdot v = \xi \otimes h(v) \cdot h|_v$, i.e., $\xi \otimes v \in K_{h|_v,n}$. By the inductive assumption, $h|_v$ belongs to the stabilizer of $\xi \otimes v$.

Consequently, $\xi \otimes h(v) = \xi \otimes v \cdot h|_v^{-1} = \xi \otimes v$. It follows that there exists $h' \in \mathcal{N}$ such that $\xi \cdot h' = \xi$, $h'h(v) = v$, and $h'|_{h(v)} = \varepsilon$. We have then for $h_1 = hh'$:

$$\xi \cdot h_1 = \xi, \quad h_1 \cdot v = h'h \cdot v = v \cdot h'|_{h(v)} h|_v = v \cdot h|_v.$$

We may assume therefore that $h(v) = v$.

Note that $h|_v$ can not belong to any of the groups of the set \mathcal{G}_1 , since then $g = h|_{vx} \in \langle \beta, b \rangle \subset \mathfrak{A}$. Consequently, $h|_v$ belongs to one of the groups of the set \mathcal{G} .

If $h|_v(x) = x$, then $\xi \otimes vx = \xi \otimes h \cdot vx = \xi \otimes vx \cdot g$, and g belongs to the stabilizer of $\xi \otimes vx$.

Suppose that $h|_v(x) \neq x$. All the generators of the groups in the set \mathcal{G} acting non-trivially on the first level belong to the set $\{\alpha, a, \alpha a\}$ of elements having trivial sections. Consequently, if $h|_v$ belongs to one of the stabilizers from \mathcal{G} , then $h|_v(x) = \delta(x)$ for $\delta \in \{\alpha, a, \alpha a\}$, and $\xi \otimes vx = \xi \otimes v \cdot \delta \otimes x = \xi \otimes v h|_v(x)$, therefore g belongs to the stabilizer of $\xi \otimes vx$. \square

Let us describe now a recursive procedure of constructing the complexes \mathcal{T}_n .

Theorem 5.4. *The space \mathcal{T}_{n+1} is the quotient of the space $\mathcal{T}_n \times \mathbf{X}$ by the equivalence relation generated by the identifications*

$$\xi \otimes x = \kappa_{g,n}(\xi) \otimes g(x)$$

for all $g \in \mathfrak{A}$, $x \in \mathbf{X}$, $\xi \in K_{g,n}$, such that $g|_x = \varepsilon$.

The set $K_{g,n+1}$ for $g \in \mathfrak{A}$ is equal to

$$\bigcup_{h \in \mathfrak{A}, x \in \mathbf{X}, h|_x = g} K_{h,n} \otimes x.$$

The map $\kappa_{g,n+1} : K_{g,n+1} \rightarrow K_{g,n+1}$ for $g \in \mathfrak{A}$ acts by the rule

$$\kappa_{g,n+1}(\xi \otimes x) = \kappa_{h,n}(\xi) \otimes h(x),$$

where $h \in \mathfrak{A}$ is such that $h|_x = g$, and $\xi \in K_{h,n}$.

All the information used in the inductive pasting rule of Theorem 5.4 is read directly from the wreath recursion (4)–(8) (Subsection 5.1) or from the structure of the automaton \mathfrak{A} on Figure 4.

For instance, the identification of the copies $\mathcal{T}_n \times x$ of \mathcal{T}_n are given by the maps:

$$\begin{aligned} (\kappa_{\alpha,n}, \sigma) &: K_{\alpha,n} \times \mathbf{X} \rightarrow K_{\alpha,n} \times \mathbf{X}, \\ (\kappa_{a,n}, \pi) &: K_{a,n} \times \mathbf{X} \rightarrow K_{a,n} \times \mathbf{X}, \\ (\kappa_{a\alpha,n}, \pi\sigma) &: K_{a\alpha,n} \times \mathbf{X} \rightarrow K_{a\alpha,n} \times \mathbf{X}, \end{aligned}$$

and

$$\begin{aligned} (\kappa_{\gamma,n}, \varepsilon) &: K_{\gamma,n} \times \{2, 3\} \rightarrow K_{\gamma,n} \times \{2, 3\}, \\ (\kappa_{a\alpha\gamma,n}, \pi\sigma) &: K_{a\alpha\gamma,n} \times \{2, 3\} \rightarrow K_{a\alpha\gamma,n} \times \{2, 3\}. \end{aligned}$$

Proof. If we replace \mathfrak{A} by \mathcal{N} everywhere in the theorem, then it will follow directly from the definition of the tensor product $\Xi_{n+1} = \Xi_n \otimes \mathfrak{M}$.

Therefore, the space \mathcal{T}_{n+1} is obtained by taking the quotient of the space $\mathcal{T}_n \times \mathbf{X}$ by the identifications

$$(9) \quad (\xi, x) \sim (\kappa_{g,n}(\xi), g(x)),$$

where $g \in \mathcal{N}$, $g|_x = \varepsilon$, and $\xi \in K_{g,n}$. Suppose that g does not belong to \mathfrak{A} . Then, by Lemma 5.3, g belongs to the stabilizer Γ_ξ of ξ , and $\Gamma_\xi \in \mathcal{G} \cup \mathcal{G}_1$. Identification (9) becomes $(\xi, x) \sim (\xi, g(x))$. If $x = g(x)$, the identification is trivial. If $g(x) \neq x$ and g is an element of one of the groups in the set \mathcal{G} , then there exists $\delta \in \{\alpha, a, a\alpha\}$ such that $\xi \cdot \delta = \xi$ and $\delta(x) = g(x)$. Then identification (9) is made using elements of \mathfrak{A} . If g is an element of a group from the set \mathcal{G}_1 , then either $g \in \mathfrak{A}$, or $g \in \{a\alpha\gamma\alpha c, ac\}$. But $a\alpha\gamma\alpha c = \pi(b\beta, b, b\beta, b)$ and $ac = \pi(b\beta, b\beta, b, b)$, which contradicts the condition $g|_x = \varepsilon$. We see that identification (9) is either trivial, or can be implemented by an element of \mathfrak{A} .

It remains to prove that every point of $K_{g,n+1}$ for $g \in \mathfrak{A}$ can be represented by $\xi \otimes x$ for $\xi \in K_{h,n}$ and $x \in \mathbf{X}$, where $h \in \mathfrak{A}$ is such that $h|_x = g$.

Every point of $K_{g,n+1}$ can be written as $\xi \otimes x$ for $\xi \in K_{h_0,n}$ and $x \in \mathbf{X}$, where $h_0 \in \mathcal{N}$ is such that $h_0|_x = g$. Suppose that $h_0 \notin \mathfrak{A}$. Then, by Lemma 5.3, h_0 belongs to the stabilizer $\Gamma_\xi \in \mathcal{G} \cup \mathcal{G}_1$. It is enough then to show that there exist $h \in \Gamma_\xi \cap \mathfrak{A}$ and $y \in \mathbf{X}$ such that $\xi \otimes x = \xi \otimes y$ and $h|_y = g$. We have $\xi \otimes x = \xi \otimes y$ if there exists $\delta \in \Gamma_\xi$ such that $\delta \cdot x = y \cdot \varepsilon$.

Thus, theorem is proved if we show that for all $G \in \mathcal{G} \cup \mathcal{G}_1$, $x \in \mathbf{X}$, and $g \in G|_x \cap \mathfrak{A}$ there exists $\delta \in G$ and $h \in G \cap \mathfrak{A}$ such that $\delta|_x = \varepsilon$ and $h|_{\delta(x)} = g$.

Let us consider all the cases. If $G = \Gamma_A = \langle \beta, \gamma, b, c \rangle$, then for every $x \in \mathbf{X}$ and $g \in G|_x \cap \mathfrak{A}$ there exists $h \in G$ such that $h|_x = g$ (so we can take $\delta = \varepsilon$):

(1) For $x = 1$, $G|_1 \cap \mathfrak{A} = \{\varepsilon, \alpha, \beta, a, b, a\alpha, b\beta\}$, and

$$\alpha = \beta|_1, \quad \beta = \gamma|_1, \quad a = (b\beta)|_1, \quad b = (c\gamma)|_1, \quad a\alpha = b|_1, \quad b\beta = c|_1.$$

(2) For $x = 2$, $G|_2 \cap \mathfrak{A} = \{\varepsilon, \gamma, a\alpha, b\beta, a\alpha\gamma\}$, and

$$\gamma = \beta|_2, \quad a\alpha = b|_2, \quad b\beta = c|_2, \quad a\alpha\gamma = (b\beta)|_2.$$

(3) For $x = 3$, $G|_3 \cap \mathfrak{A} = \{\varepsilon, \alpha, b, c, \alpha c\}$, and

$$\alpha = \beta|_3, \quad b = c|_3, \quad c = b|_3, \quad \alpha c = (b\beta)|_3.$$

(4) For $x = 4$, $G|_4 \cap \mathfrak{A} = \{\varepsilon, \beta, \gamma, b, c, b\beta, c\gamma\}$, and

$$\beta = \gamma|_4, \quad \gamma = \beta|_4, \quad b = c|_4, \quad c = b|_4, \quad b\beta = (c\gamma)|_4, \quad c\gamma = (b\beta)|_4.$$

If $G = \Gamma_B = \langle \alpha, \gamma, a, c \rangle$, then $G|_x = \langle \beta, b \rangle$ for all $x \in \mathbf{X}$. For any pair $x, y \in \mathbf{X}$ there exists $\delta \in \langle \alpha, a \rangle \subset \Gamma_B$ such that $\delta(x) = y$. Hence, equalities

$$\beta = \gamma|_1, \quad b = c|_1, \quad b\beta = (c\gamma)|_1,$$

finish the proof for $G = \Gamma_B$.

The case $G = \Gamma_C = \langle \alpha, \beta, a, b \rangle$ is considered in the same way. We have $G|_x = \langle \alpha, \gamma, a, c \rangle$ for all $x \in \mathbf{X}$, and

$$\begin{aligned} \alpha &= \beta|_1, \quad a = (b\beta)|_1, \quad a\alpha = b|_1, \\ \gamma &= \beta|_2, \quad c = b|_3, \quad c\gamma = (b\beta)|_4, \\ \alpha c &= (b\beta)|_3, \quad a\alpha\gamma = (b\beta)|_2. \end{aligned}$$

Consider the case $G = \Gamma_{A_1} = \langle \alpha, b, c \rangle$. If $x \in \{1, 2\}$, then $G|_x \cap \mathfrak{A} = \{\varepsilon, a\alpha, b\beta\}$, and

$$a\alpha = b|_x, \quad b\beta = c|_x.$$

If $x \in \{3, 4\}$, then $G|_x \cap \mathfrak{A} = \{\varepsilon, b, c\}$ and

$$b = c|_x, \quad c = b|_x.$$

Cases $G = \Gamma_{B_1}$ and $G = \Gamma_{C_1}$ are similar to Γ_{A_1} .

Cases $G \in \{\Gamma_{AB}, \Gamma_{AC}, \Gamma_{AA_1}, \Gamma_{AB_1}, \Gamma_{AC_1}\}$ are straightforward: G acts trivially on the first level; $G|_x \cap \mathfrak{A}$ coincides with the standard generating set; and for every $x \in \mathbf{X}$, $g \in G|_x \cap \mathfrak{A}$ there exists $h \in G \cap \mathfrak{A}$ such that $h|_x = g$.

There is nothing to prove for $G = \Gamma_{BC}$.

Cases $G \in \{\Gamma_{BA_1}, \Gamma_{BB_1}, \Gamma_{BC_1}, \Gamma_{CA_1}, \Gamma_{CB_1}, \Gamma_{CC_1}\}$ are similar to $G = \Gamma_{A_1}$: the group G contains an element $\delta \in \{\alpha, a, a\alpha\}$; intersection of $G|_x$ with \mathfrak{A} is the standard generating set of $G|_x$; for every $x \in \mathbf{X}$ and $g \in G|_x \cap \mathfrak{A}$ there exists $h \in G \cap \mathfrak{A}$ such that either $h|_x = g$, or $h|_{\delta(x)} = g$.

If $G = \langle g \rangle \in \mathcal{G} \cup \mathcal{G}_1$ is cyclic, then $g, g|_x \in \mathfrak{A}$, and we are done.

The only remaining case is $G = \langle a\alpha\gamma, \alpha c \rangle$. Since $a\alpha\gamma \cdot \alpha c = \pi(b\beta, b, b\beta, b) = \gamma ac$ is an involution, we have

$$G = \{\varepsilon, \quad a\alpha\gamma = \pi\sigma(\beta, \varepsilon, \varepsilon, \beta), \quad \alpha c = \sigma(b\beta, b\beta, b, b), \quad \gamma ac = \pi(b\beta, b, b\beta, b)\},$$

hence

$$G|_1 = \{\varepsilon, \beta = (a\alpha\gamma)|_1, b\beta = (\alpha c)|_1\}, \quad G|_4 = \{\varepsilon, \beta = (a\alpha\gamma)|_4, b = (\alpha c)|_4\}.$$

We also have $a\alpha\gamma \cdot 2 = 3 \cdot \varepsilon$, $a\alpha\gamma \cdot 3 = 2 \cdot \varepsilon$, and

$$G|_2 = G|_3 = \{\varepsilon, b = (\alpha c)|_2, b = (\alpha c)|_3\},$$

which finishes the proof. \square

5.6. Equivariant map. Let us construct the complex \mathcal{T}_1 . By Theorem 5.4, it is obtained by gluing four copies $\mathcal{T}_0 \otimes x$, for $x = 1, 2, 3, 4$, of $\mathcal{T}_0 = \mathcal{M}$ along the following faces:

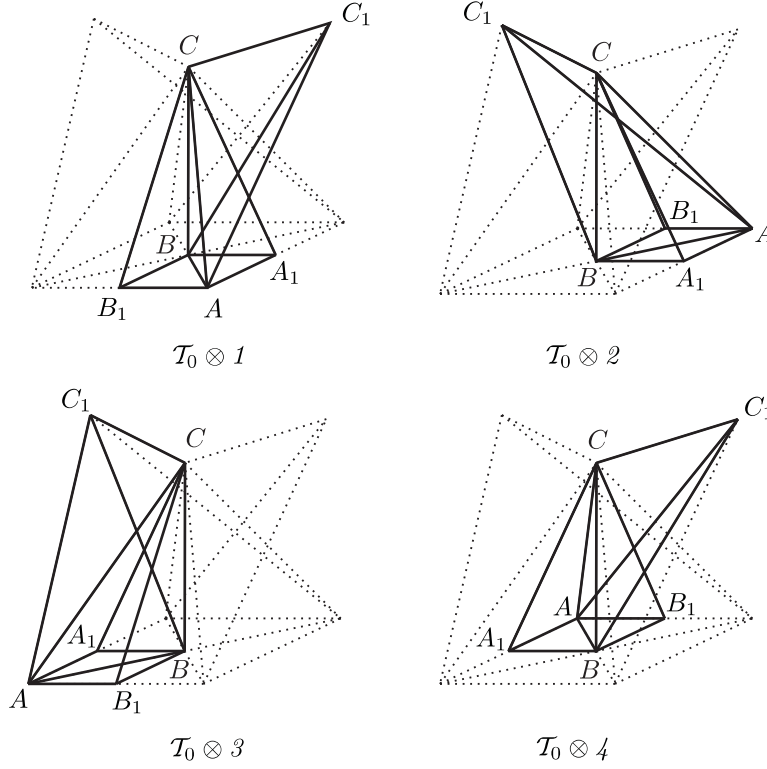
$$\begin{aligned} (\kappa_{\alpha,0}, \sigma) : A_1BC \times 1 &\sim A_1BC \times 2, \quad A_1BC \times 3 \sim A_1BC \times 4, \\ (\kappa_{a,0}, \pi) : B_1BC \times 1 &\sim B_1BC \times 3, \quad B_1BC \times 2 \sim B_1BC \times 4, \\ (\kappa_{a\alpha,0}, \pi\sigma) : C_1BC \times 1 &\sim C_1BC \times 4, \quad C_1BC \times 2 \sim C_1BC \times 3, \end{aligned}$$

where in each case the identification is identical on the first coordinate. Note that the identification $(\kappa_{a\alpha\gamma,0}, \pi\sigma)$ of $C_1B \times 2$ with $C_1B \times 3$ follows from the identification of $C_1BC \times 2$ with $C_1BC \times 3$. The identification $(\kappa_{\gamma,0}, \varepsilon)$ is trivial.

The resulting complex \mathcal{T}_1 consists of a square pyramid and two tetrahedra such that one face of each tetrahedron is attached to a diagonal of the square pyramid (see Figure 7, but ignore the labels of the vertices this time). Figure 6 shows the parts $\mathcal{T}_0 \otimes i$ of \mathcal{T}_1 (the vertices of each part $\mathcal{T}_0 \otimes i$ are labeled as in \mathcal{T}_0).

The map $\xi \otimes x \mapsto \xi$ folds the square pyramid in four using reflections with respect to the planes passing through the point $C \otimes 1 = C \otimes 2 = C \otimes 3 = C \otimes 4$ and through midpoints of two opposite sides of the base of the pyramid, and folds the two tetrahedra along the planes passing through the images of $C_1 \otimes i$ and the height of the pyramid.

The elements of \mathfrak{A} acting non-trivially on the first level are $\alpha, a, a\alpha, \alpha c$, and $a\alpha\gamma$. The first three elements have only trivial sections, hence they produce only the identifications of the copies of \mathcal{T}_0 and no maps $\kappa_{g,1}$.

FIGURE 6. Pasting together \mathcal{T}_1

The element $\alpha c = \sigma(b\beta, b\beta, b, b)$ produces a part of the map $\kappa_{b\beta,1}$ switching $K_{\alpha c,0} \times 1$ with $K_{\alpha c,0} \times 2$ and a part of $\kappa_{b,1}$ switching $K_{\alpha c,0} \times 3$ with $K_{\alpha c,0} \times 4$. But $K_{\alpha c,0}$ is the segment A_1B , for which we have identifications $A_1B \times 1 \sim A_1B \times 2$ and $A_1B \times 3 \sim A_1B \times 4$ as parts of the identifications $A_1BC \times i \sim A_1BC \times \sigma(i)$.

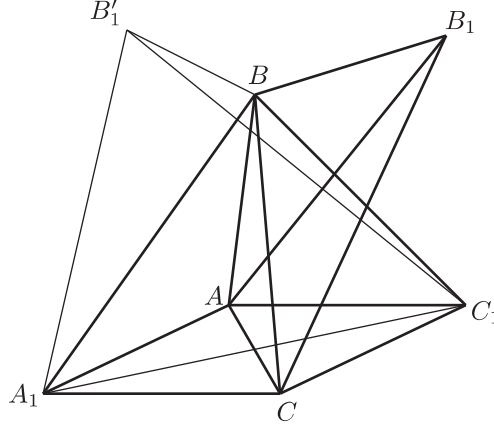
The element $a\alpha\gamma = \pi\sigma(\beta, \varepsilon, \varepsilon, \beta)$ produces a part of the map $\kappa_{\beta,1}$ switching $K_{a\alpha\gamma,0} \times 1 = C_1B \times 1$ with $K_{a\alpha\gamma,0} \times 3 = C_1B \times 3$. But these sets are also identified in \mathcal{T}_1 .

We see that all the maps $\kappa_{g,1}$ are identical. Their domains are

$$\begin{aligned}
 K_{\alpha,1} &= B_1AC \otimes \{1, 3\}, & K_{\beta,1} &= C_1AB \otimes \{1, 4\}, & K_{\gamma,1} &= B_1AC \otimes \{2, 4\} \\
 K_{a,1} &= C_1CA \otimes 1, & K_{c,1} &= A_1AC \otimes \{3, 4\}, \\
 K_{b,1} &= A_1AB \otimes \{3, 4\} \cup B_1BA \otimes \{1, 3\}, \\
 K_{a\alpha,1} &= A_1AC \otimes \{1, 2\}, & K_{c\gamma,1} &= C_1CA \otimes 4 \\
 K_{b\beta,1} &= A_1AB \otimes \{1, 2\} \cup B_1BA \otimes \{2, 4\}, \\
 K_{a\alpha\gamma,1} &= C_1CA \otimes 2, & K_{\alpha c,1} &= C_1CA \otimes 3.
 \end{aligned}$$

We have seen in Subsection 5.4 that for every $G \in \mathcal{G}$ and every $x \in \mathbf{X}$ there exists $H \in \mathcal{G}$ such that $G|_x \subseteq H$.

For $G \in \{\Gamma_A, \Gamma_B, \Gamma_C, \Gamma_{A_1}, \Gamma_{B_1}, \Gamma_{C_1}\}$ and for $x \in \mathbf{X}$ denote by $I(G, x)$ the intersection of the groups $H \in \mathcal{G}$ for which $G|_x \subseteq H$.

FIGURE 7. The complex \mathcal{T}_1

Denote for $G \in \{\Gamma_A, \Gamma_B, \Gamma_C, \Gamma_{A_1}, \Gamma_{B_1}, \Gamma_{C_1}\}$ and $g \in \Gamma$:

$$I(G \cdot g, x) = I(G, g(x)) \cdot g|_x.$$

Then $I(G \cdot g, x)$ is the intersection of the cosets $H \cdot h$, for $H \in \mathcal{G}$ and $h \in \Gamma$, containing the set $(G \cdot g)|_x$. The maps $I(\cdot, x)$ satisfy the condition

$$I(U \cdot g, x) = I(U, g(x)) \cdot g|_x,$$

for all cosets U and for all $g \in \Gamma$, $x \in \mathbf{X}$.

Recall that $\Xi = \bigcup_{g \in \Gamma} \mathcal{T}_0 \cdot g$, where \mathcal{T}_0 is a union of three tetrahedra A_1ABC , B_1ABC , C_1ABC with vertices corresponding to the groups $\Gamma_A, \Gamma_B, \Gamma_C, \Gamma_{A_1}, \Gamma_{B_1}$, and Γ_{C_1} . It is checked directly that for every $x \in \mathbf{X}$ the image of the set of vertices of each of these tetrahedra under $I(\cdot, x)$ is a subset of one of the tetrahedra (we will see this also in the geometric description of the maps $I(\cdot, x)$ below). Consequently, we can extend $I(\cdot, x)$ by linearity to the whole complex Ξ . In this way we get continuous maps satisfying the condition

$$I(\xi \cdot g, x) = I(\xi, g(x)) \cdot g|_x,$$

for all $\xi \in \Xi$, $g \in \Gamma$, and $x \in \mathbf{X}$. Hence, the map $I(\xi \otimes x) = I(\xi, x)$ is a well defined continuous equivariant map from $\Xi_1 = \Xi \otimes \mathfrak{M}$ to Ξ .

Figure 7 shows the complex \mathcal{T}_1 in the same way as it is shown on Figure 6, but with vertices labeled by their images under the map I (except for B'_1 , which is mapped to B). One tetrahedron ($B'_1A_1BC_1$ on Figure 7) is collapsed by I onto the diagonal of the square pyramid (triangle A_1BC_1). The remaining part of \mathcal{T}_1 is a union of three tetrahedra and is mapped by a locally affine homeomorphism onto \mathcal{T}_0 .

We have the following formulae for I (see Figures 5):

$$\begin{aligned} I(A \otimes 1) &= C, & I(B \otimes 1) &= (AC), & I(C \otimes 1) &= B, \\ I(A \otimes 2) &= C_1, & I(B \otimes 2) &= (AC), & I(C \otimes 2) &= B, \\ I(A \otimes 3) &= A_1, & I(B \otimes 3) &= (AC), & I(C \otimes 3) &= B, \\ I(A \otimes 4) &= A, & I(B \otimes 4) &= (AC), & I(C \otimes 4) &= B, \end{aligned}$$

and

$$\begin{aligned} I(A_1 \otimes 1) &= (CC_1), \quad I(B_1 \otimes 1) = (CA_1), \quad I(C_1 \otimes 1) = B_1, \\ I(A_1 \otimes 2) &= (CC_1), \quad I(B_1 \otimes 2) = (AC_1), \quad I(C_1 \otimes 2) = B, \\ I(A_1 \otimes 3) &= (AA_1), \quad I(B_1 \otimes 3) = (CA_1), \quad I(C_1 \otimes 3) = B, \\ I(A_1 \otimes 4) &= (AA_1), \quad I(B_1 \otimes 4) = (AC_1), \quad I(C_1 \otimes 4) = B_1, \end{aligned}$$

where (XY) denotes the midpoint of the segment XY . The vertices $X \otimes i$ are shown on Figure 6: a vertex $X \otimes i$ is labeled by X on the part showing $\mathcal{T}_0 \otimes i$.

The map $I : \Xi \otimes \mathfrak{M} \rightarrow \Xi$ is uniquely determined by its restriction $I : \mathcal{T}_1 \rightarrow \mathcal{T}_0$ onto the fundamental domain, due to equivariance.

Let us introduce a Euclidean structure on the complex Ξ by embedding the complex \mathcal{T}_0 into \mathbb{R}^5 in such a way that

$$\{\overrightarrow{BA}, \overrightarrow{BC}, \overrightarrow{AA_1}, \overrightarrow{BB_1}, \overrightarrow{CC_1}\}$$

is the standard orthonormal basis of \mathbb{R}^6 .

Proposition 5.5. *The map $I : \Xi \otimes \mathfrak{M} \rightarrow \Xi$ is contracting.*

Proof. The map $\xi \mapsto I(\xi \otimes x)$ is affine on \mathcal{T}_0 for every $x \in \mathbf{X}$. Let \mathcal{I}_x be its linear part.

We have

$$\begin{aligned} \mathcal{I}_1(\overrightarrow{BA}) &= \frac{1}{2}\overrightarrow{AC} = -\frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{BC}, \quad \mathcal{I}_1(\overrightarrow{BC}) = -\frac{1}{2}\overrightarrow{BA} - \frac{1}{2}\overrightarrow{BC}, \\ \mathcal{I}_1(\overrightarrow{AA_1}) &= \frac{1}{2}\overrightarrow{CC_1}, \quad \mathcal{I}_1(\overrightarrow{BB_1}) = \frac{1}{2}\overrightarrow{AA_1}, \quad \mathcal{I}_1(\overrightarrow{CC_1}) = \overrightarrow{BB_1}, \end{aligned}$$

hence

$$\mathcal{I}_1 = \begin{pmatrix} -1/2 & -1/2 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 0 & 0 \end{pmatrix}.$$

We have

$$\begin{aligned} \mathcal{I}_2(\overrightarrow{BA}) &= \frac{1}{2}\overrightarrow{AC_1} + \frac{1}{2}\overrightarrow{CC_1} = -\frac{1}{2}\overrightarrow{BA} + \frac{1}{2}\overrightarrow{BC} + \overrightarrow{CC_1} \\ \mathcal{I}_2(\overrightarrow{BC}) &= \frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{CB} = -\frac{1}{2}\overrightarrow{BA} - \frac{1}{2}\overrightarrow{BC} \\ \mathcal{I}_2(\overrightarrow{AA_1}) &= -\frac{1}{2}\overrightarrow{CC_1}, \quad \mathcal{I}_2(\overrightarrow{BB_1}) = \frac{1}{2}\overrightarrow{CC_1}, \quad \mathcal{I}_2(\overrightarrow{CC_1}) = \overrightarrow{0}, \end{aligned}$$

hence

$$\mathcal{I}_2 = \begin{pmatrix} -1/2 & -1/2 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1/2 & 1/2 & 0 \end{pmatrix}.$$

The map \mathcal{I}_3 acts by

$$\begin{aligned} \mathcal{I}_3(\overrightarrow{BA}) &= \frac{1}{2}\overrightarrow{AA_1} + \frac{1}{2}\overrightarrow{CA_1} = \frac{1}{2}\overrightarrow{BA} - \frac{1}{2}\overrightarrow{BC} + \overrightarrow{AA_1}, \quad \mathcal{I}_3(\overrightarrow{BC}) = -\frac{1}{2}\overrightarrow{BA} - \frac{1}{2}\overrightarrow{BC}, \\ \mathcal{I}_3(\overrightarrow{AA_1}) &= -\frac{1}{2}\overrightarrow{AA_1}, \quad \mathcal{I}_3(\overrightarrow{BB_1}) = \frac{1}{2}\overrightarrow{AA_1}, \quad \mathcal{I}_3(\overrightarrow{CC_1}) = \overrightarrow{0}, \end{aligned}$$

hence

$$\mathcal{I}_3 = \begin{pmatrix} 1/2 & -1/2 & 0 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 & 0 \\ 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally,

$$\begin{aligned} \mathcal{I}_4(\overrightarrow{BA}) &= \frac{1}{2}\overrightarrow{CA} = \frac{1}{2}\overrightarrow{BA} - \frac{1}{2}\overrightarrow{BC}, & \mathcal{I}_4(\overrightarrow{BC}) &= -\frac{1}{2}\overrightarrow{BA} - \frac{1}{2}\overrightarrow{BC}, \\ \mathcal{I}_4(\overrightarrow{AA_1}) &= \frac{1}{2}\overrightarrow{AA_1}, & \mathcal{I}_4(\overrightarrow{BB_1}) &= \frac{1}{2}\overrightarrow{CC_1}, & \mathcal{I}_4(\overrightarrow{CC_1}) &= \overrightarrow{BB_1}, \end{aligned}$$

hence

$$\mathcal{I}_4 = \begin{pmatrix} 1/2 & -1/2 & 0 & 0 & 0 \\ -1/2 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1/2 & 0 \end{pmatrix}.$$

We see that all matrices \mathcal{I}_x are of the block-triangular form $\left(\begin{array}{c|c} U_x & 0 \\ \hline W_x & V_x \end{array} \right)$, where U_x and V_x are of size 2×2 and 3×3 , respectively. For every vector \vec{v} and every $x \in \mathbb{X}$ the Euclidean length $\|U_x \vec{v}\|$ is equal to $\|\vec{v}\|/\sqrt{2}$. Consequently, the norm of any product $U_{x_1}U_{x_2}\cdots U_{x_n}$ of length n is equal to $2^{-n/2}$.

It is straightforward to check that for any two indices $x_1, x_2 \in \mathbb{X}$ the norm of $V_{x_1}V_{x_2}$ is not more than $1/\sqrt{2}$. Consequently, there is a constant C such that norm of $V_{x_1}V_{x_2}\cdots V_{x_n}$ is not more than $C2^{-n/4}$. Norm of W_x does not exceed 1.

The product $\mathcal{I}_{x_1}\mathcal{I}_{x_2}\cdots\mathcal{I}_{x_n}$ is of the form $\left(\begin{array}{c|c} U & 0 \\ \hline W & V \end{array} \right)$, where

$$U = U_{x_1}U_{x_2}\cdots U_{x_n}, \quad V = V_{x_1}V_{x_2}\cdots V_{x_n},$$

and

$$W = \sum_{k=1}^n V_{x_1}\cdots V_{x_{k-1}}W_{x_k}U_{x_{k+1}}\cdots U_{x_n}.$$

The norm of W is estimated then as follows

$$\begin{aligned} \|W\| &\leq \sum_{k=1}^n \|V_{x_1}\cdots V_{x_{k-1}}\| \cdot \|W_{x_k}\| \cdot \|U_{x_{k+1}}\cdots U_{x_n}\| \leq \\ &\sum_{k=1}^n C2^{-k/4} \cdot 2^{-(n-k)/2} \leq \sum_{k=1}^n C2^{-k/4} \cdot 2^{-(n-k)/4} = nC2^{-n/4}. \end{aligned}$$

It follows that the norm of the product $\mathcal{I}_{x_1}\mathcal{I}_{x_2}\cdots\mathcal{I}_{x_n}$ uniformly converges to 0 as n goes to infinity. Consequently, there exists n such that the map $\xi \mapsto I^{(n)}(\xi \otimes v)$ contracts all distances in Ξ at least by $1/2$ for all $v \in \mathbb{X}^n$. \square

5.7. Complexes approximating the Julia set. Recall (see 5.1) that $\text{IMG}(f)$ is the index two subgroup of Γ generated by $\alpha, \beta, \gamma, ab, bc$. Then Ξ is also a co-compact proper $\text{IMG}(f)$ -space.

Let \mathfrak{M} and \mathfrak{M}_f be the self-similarity bimodules of Γ and $\text{IMG}(f)$, respectively. Since $\text{IMG}(f)$ is a subgroup of Γ , the bimodule \mathfrak{M}_f is a subset of \mathfrak{M} . Let $\mathbb{Y} =$

$\{1, 2, 3 \cdot b, 4 \cdot b\}$ be the common basis of these bimodules, corresponding to the wreath recursion in Theorem 4.3 defining $\text{IMG}(f)$.

Lemma 5.6. *The identical map $\Xi \times Y^n \rightarrow \Xi \times Y^n$ induces a homeomorphism $\Xi \otimes_{\Gamma} \mathfrak{M}^{\otimes n} \rightarrow \Xi \otimes_{\text{IMG}(f)} \mathfrak{M}_f^{\otimes n}$.*

Proof. Every element of $\Xi_n = \Xi \otimes_{\Gamma} \mathfrak{M}^{\otimes n}$ can be represented by $(\xi, 1^n)$ for some $\xi \in \Xi$, since the group Γ is self-replicating (i.e., the left action of Γ on \mathfrak{M} , and hence on $\mathfrak{M}^{\otimes n}$, is transitive). The same is true for $\Xi \otimes_{\text{IMG}(f)} \mathfrak{M}_f$. Two pairs $(\xi_1, 1^n)$ and $(\xi_2, 1^n)$ represent the same point of $\Xi \otimes_{\Gamma} \mathfrak{M}^{\otimes n}$ (resp. of $\Xi \otimes_{\text{IMG}(f)} \mathfrak{M}_f^{\otimes n}$) if and only if there exists $g \in \Gamma$ (resp. $g \in \text{IMG}(f)$) such that

$$\xi_1 \cdot g = \xi_2, \quad g(1^n) = 1^n, \quad g|_{1^n} = \varepsilon.$$

Denote by K the kernel of the virtual endomorphism of Γ associated with the word $1^n \in X^n$ (i.e., the subgroup of the elements of Γ such that $g(1^n) = 1^n$ and $g|_{1^n} = \varepsilon$). It is sufficient to prove that $K < \text{IMG}(f)$.

It follows from [Nek07, Proposition 4.7] that if a product of the generators $\alpha, \beta, \gamma, a, b, c$ is trivial in Γ , then the numbers of occurrences of each of the letters a, b, c are even. Consequently, a product of the generators of Γ is an element of $\text{IMG}(f)$ if and only if the total number of occurrences of the letters a, b, c is even.

It follows from the wreath recursion defining Γ that the parity of the total number of occurrences of the letters a, b, c in g is the same as in $g|_1$, if $g(1) = 1$. It follows that if $g(1^n) = 1^n$, then the total number of occurrences of the letters a, b, c in g is the same as in $g|_{1^n}$. Consequently, if $g|_{1^n} = \varepsilon$, then $g \in \text{IMG}(f)$. \square

As a corollary of the lemma, we get that the map $I : \Xi \otimes \mathfrak{M} \rightarrow \Xi$ can be seen as an $\text{IMG}(f)$ -equivariant map $I : \Xi \otimes \mathfrak{M}_f \rightarrow \Xi$, and that the induced maps $I_n : \Xi \otimes \mathfrak{M}_f^{\otimes(n+1)} \rightarrow \Xi \otimes \mathfrak{M}_f^{\otimes n}$ are the same as the maps $I_n : \Xi \otimes \mathfrak{M}^{\otimes(n+1)} \rightarrow \Xi \otimes \mathfrak{M}^{\otimes n}$.

Consequently, just restricting our construction to the index two subgroup $\text{IMG}(f)$ of Γ we get approximations $\mathcal{M}_n = \Xi_n / \text{IMG}(f)$ of the limit space $\mathcal{J}_{\text{IMG}(f)}$. Namely, the following theorem follows directly from Corollary 3.5.

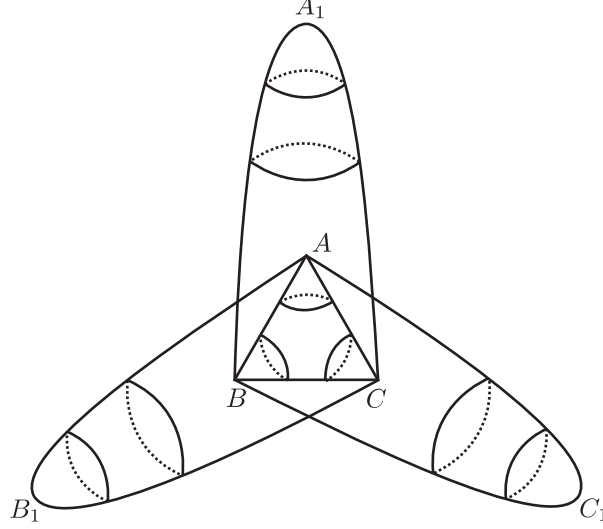
Theorem 5.7. *Let $\iota_n : \mathcal{M}_{n+1} \rightarrow \mathcal{M}_n$ and $p_n : \mathcal{M}_{n+1} \rightarrow \mathcal{M}_n$ be the maps induced by $I_n : \Xi_{n+1} \rightarrow \Xi_n$ and the correspondence $\xi \otimes x \mapsto \xi$ for $x \in \mathfrak{M}_f$, respectively. Let p_∞ be the map induced by the maps p_n on the inverse limit \mathcal{M}_∞ of the sequence*

$$\mathcal{M}_0 \xleftarrow{\iota_0} \mathcal{M}_1 \xleftarrow{\iota_1} \mathcal{M}_2 \xleftarrow{\iota_2} \dots$$

Then the dynamical system $(\mathcal{M}_\infty, p_\infty)$ is topologically conjugate to the limit dynamical system of $\text{IMG}(f)$, which is conjugate to the action of f on its Julia set (if f is sub-hyperbolic).

The approximations \mathcal{M}_n of the limit space $\mathcal{J}_{\text{IMG}(f)}$ (i.e., of the Julia set of f , if Conjecture from Section 2 is true) can be constructed from the complexes \mathcal{T}_n in the following way.

Proposition 5.8. *Let \mathcal{T}_n and $\kappa_{g,n}$ be as in Theorem 5.4. The complex \mathcal{M}_n is obtained by pasting two copies \mathcal{T}_n and $\mathcal{T}_n \cdot a$ of \mathcal{T}_n along the sets $K_{a,n}, K_{b,n}, K_{c,n}, K_{a\alpha,n}, K_{b\beta,n}, K_{c\gamma,n}, K_{a\alpha\gamma,n}$, and $K_{ac,n}$ by the action of the respective maps $\kappa_{g,n}$ (i.e., identifying a point ξ of one copy with the point $\kappa_{g,n}(\xi) \cdot a$ in the other copy) and pasting the sets $K_{\alpha,n}, K_{\beta,n}$, and $K_{\gamma,n}$ to themselves (inside each of the copies) by the respective $\kappa_{g,n}$.*

FIGURE 8. Complex $\mathcal{M} = \Xi / \text{IMG}(f)$

Proof. The set $\mathcal{T}_0 \cup \mathcal{T}_0 \cdot a \subset \Xi$ is a fundamental domain of the action of $\text{IMG}(f)$ on Ξ . Consequently, the orbispace \mathcal{M}_n is obtained by identifying in the union $\mathcal{T}_0 \cup \mathcal{T}_0 \cdot a$ any two points belonging to one $\text{IMG}(f)$ -orbit. Two different points $\xi_1, \xi_2 \in \mathcal{T}_0 \cdot \varepsilon$ belong to one $\text{IMG}(f)$ -orbit if and only if there exists $g \in \mathcal{N} \cap \text{IMG}(f)$ such that $\xi_1 \cdot g = \xi_2$. By Lemma 5.3, g belongs to \mathfrak{A} . But $\{\alpha, \beta, \gamma\}$ are the only elements of $\mathfrak{A} \cap \text{IMG}(f)$. Two points $\xi_1 \cdot a, \xi_2 \cdot a \in \mathcal{T}_0 \cdot a$ belong to one $\text{IMG}(f)$ -orbit if and only if $\xi_1, \xi_2 \in \mathcal{T}_0$ belong to one $\text{IMG}(f)$ -orbit. We have proved that two points inside one of the copies of \mathcal{T}_0 are identified in \mathcal{M}_n if and only if they are either equal or are obtained from each other by application of one of the transformations $\kappa_{\alpha,n}, \kappa_{\beta,n}$, or $\kappa_{\gamma,n}$.

Suppose that $\xi_1 \in \mathcal{T}_0 \cdot \varepsilon$ and $\xi_2 \cdot a \in \mathcal{T}_0 \cdot a$ belong to one $\text{IMG}(f)$ -orbit, i.e., that $\xi_1 \cdot g = \xi_2 \cdot a$ for some $g \in \text{IMG}(f)$.

If $\xi_1 = \xi_2$, then ga belongs to the stabilizer Γ_{ξ_1} of ξ_1 , which by Lemma 5.3 is generated by elements of \mathfrak{A} . Since $ga \notin \text{IMG}(f)$, one of these generators h does not belong to $\text{IMG}(f)$. Then $\xi_1 = \xi_2 \cdot h$ for $h \in \mathfrak{A} \setminus \text{IMG}(f)$.

If $\xi_1 \neq \xi_2$, then $h = ga \in \mathfrak{A}$ by Lemma 5.3, and we again have $\xi_1 = \xi_2 \cdot h$ for $h \in \mathfrak{A} \setminus \text{IMG}(f)$. Consequently, two points belonging to different copies of \mathcal{T}_0 are identified by transformations $\kappa_{g,n}$ for $g \in \mathfrak{A} \setminus \text{IMG}(f)$. \square

In particular, the space $\mathcal{M} = \Xi / \text{IMG}(f)$ is obtained by taking two copies \mathcal{T}_0 and $\mathcal{T}_0 \cdot a$ of \mathcal{T}_0 and pasting them together by the maps $\xi \mapsto \xi \cdot a$ along $A_1AB, A_1AC, B_1BA, B_1BC, C_1CA, C_1CB$. We get in this way three solid balls (with surfaces equal to doubles of the faces A_1BC, AB_1C and ABC_1) with a common spherical hole (whose surface is double of the triangle ABC). See a schematic diagram of the complex on Figure 8.

The covering $p_n : \mathcal{M}_{n+1} \rightarrow \mathcal{M}_n$ is induced by the correspondence $(\xi \otimes x) \mapsto \xi$, where $\xi \in \Xi_n = \Xi \otimes \mathfrak{M}_f^{\otimes n}$ and $x \in \mathfrak{M}_f$.

We will denote by $[\xi]$ the image of $\xi \in \Xi_n$ in \mathcal{M}_n , i.e., the $\text{IMG}(f)$ -orbit of ξ .

Proposition 5.9. *The covering $p_n : \mathcal{M}_{n+1} \rightarrow \mathcal{M}_n$ acts by the rule*

$$\begin{aligned} p_n([\xi \otimes 1]) &= [\xi], & p_n([\xi \otimes 2]) &= [\xi], \\ p_n([\xi \otimes 3]) &= [\xi \cdot a], & p_n([\xi \otimes 4]) &= [\xi \cdot a], \end{aligned}$$

and

$$\begin{aligned} p_n([\xi \otimes 1 \cdot a]) &= [\xi \cdot a], & p_n([\xi \otimes 2 \cdot a]) &= [\xi \cdot a], \\ p_n([\xi \otimes 3]) &= [\xi], & p_n([\xi \otimes 4]) &= [\xi]. \end{aligned}$$

Proof. We have for $\xi \in \Xi_n$:

$$\xi \otimes 3 = \xi \cdot c \otimes 3 \cdot b,$$

hence $p_n([\xi \otimes 3]) = [\xi \cdot c] = [\xi \cdot a]$. Similarly,

$$\xi \otimes 4 = \xi \cdot c \otimes 4 \cdot b,$$

implies that $p_n([\xi \otimes 4]) = [\xi \cdot a]$.

Since

$$\xi \otimes 1 \cdot a = \xi \cdot b\beta \otimes 1,$$

$p_n([\xi \otimes 1 \cdot a]) = [\xi \cdot b\beta] = [\xi \cdot a]$. Similarly, since

$$\xi \otimes 2 \cdot a = \xi \cdot b\alpha\beta\alpha \otimes 2,$$

$p_n([\xi \otimes 2 \cdot a]) = [\xi \cdot b\alpha\beta\alpha] = [\xi \cdot a]$.

Equalities

$$\xi \otimes 3 \cdot a = \xi \otimes 3 \cdot b \cdot ba, \quad \xi \otimes 4 \cdot a = \xi \otimes 4 \cdot b \cdot ba$$

show that $p_n([\xi \otimes i \cdot a]) = [\xi]$ for $i \in \{3, 4\}$. \square

Equivariance of $I_n : \Xi_{n+1} \rightarrow \Xi_n$ and the fact that $I_n(\mathcal{T}_{n+1}) = \mathcal{T}_n$ imply that for any $\xi \in \mathcal{T}_{n+1}$

$$\iota_n([\xi]) = [I_n(\xi)], \quad \iota_n([\xi \cdot a]) = [I_n(\xi) \cdot a].$$

5.8. Spaces \mathcal{M}_n as subsets of the Julia set. Restriction of the map $I : \mathcal{T}_1 \rightarrow \mathcal{T}_0$ onto closure of $\mathcal{T}_1 \setminus (C_1ABC \otimes \{2, 3\})$ is a homeomorphism (see Figures 6, 7 and definition of I in Subsection 5.6). Let $\Theta : \mathcal{T}_0 \rightarrow \mathcal{T}_1$ be its inverse.

It is checked directly that for every $g \in \mathfrak{A}$ we have $\Theta(K_{g,0}) \subseteq K_{g,1}$. Suppose that $\xi \cdot g_1 = \xi \cdot g_2$ for $g_1, g_2 \in \Gamma$ and $\xi \in \mathcal{T}_0$. Then $\xi = \xi \cdot g_2 g_1^{-1}$, hence $g_2 g_1^{-1}$ belongs to the stabilizer Γ_ξ of ξ . The stabilizer of $\Theta(\xi)$ contains Γ_ξ , by Lemma 5.3, since $\Theta(K_{g,0}) \subseteq K_{g,1}$ and all transformations $\kappa_{g,1}$ act trivially. Consequently, $\Theta(\xi) \cdot g_1 = \Theta(\xi) \cdot g_2$.

We have proved that Θ can be extended by the rule $\Theta(\xi \cdot g) = \Theta(\xi) \cdot g$ to a Γ -equivariant map $\Theta : \Xi \rightarrow \Xi \otimes \mathfrak{M}$. It will be a section of the map $I : \Xi \otimes \mathfrak{M} \rightarrow \Xi$, i.e., $I \circ \Theta : \Xi \rightarrow \Xi$ is identical.

Define

$$\Theta_n(\xi \otimes v) = \Theta(\xi) \otimes v$$

for $\xi \in \Xi$ and $v \in \mathfrak{M}^{\otimes n}$. The map Θ_n is well defined by equivariance of Θ . We get hence a sequence $\Theta_n : \Xi_n \rightarrow \Xi_{n+1}$ of sections of the maps $I_n : \Xi_{n+1} \rightarrow \Xi_n$.

Denote by $\theta_n : \mathcal{M}_n \rightarrow \mathcal{M}_{n+1}$ the maps induced by Θ_n on the orbispaces $\mathcal{M}_n = \Xi_n / \text{IMG}(f)$. The maps θ_n are sections of the maps $\iota_n : \mathcal{M}_{n+1} \rightarrow \mathcal{M}_n$.

We get hence natural homeomorphism $\tilde{\theta}_n$ of the space \mathcal{M}_n with a subset of the inverse limit $\mathcal{M}_\infty \approx \mathcal{I}_{\text{IMG}(f)}$. It is the limit of the maps $\theta_{n+k} \circ \theta_{n+k-1} \circ \cdots \circ \theta_n : \mathcal{M}_n \rightarrow \mathcal{M}_{n+k+1}$ as $k \rightarrow \infty$.

The next theorem is now straightforward.

Theorem 5.10. *Let $\tilde{\theta}_n : \mathcal{M}_n \rightarrow \mathcal{M}_\infty$ be the limit of the maps $\theta_{n+k} \circ \dots \circ \theta_n$. Then $\tilde{\theta}_{n+1}(\mathcal{M}_{n+1}) \supset \tilde{\theta}_n(\mathcal{M}_n)$, $p_\infty^{-1}(\tilde{\theta}_n(\mathcal{M}_n)) = \tilde{\theta}_{n+1}(\mathcal{M}_{n+1})$, $p_n = \tilde{\theta}_n^{-1} \circ p_\infty \circ \tilde{\theta}_{n+1}$, and the set $\bigcup_{n \geq 1} \tilde{\theta}_n(\mathcal{M}_n)$ is dense in \mathcal{M}_∞ .*

We will give later a natural description of the sets $\tilde{\theta}_n(\mathcal{M}_n)$ as subsets of the Julia set of f .

6. SKEW PRODUCT DECOMPOSITION

6.1. The projection $(z, w) \mapsto w$. Projection $(z, w) \mapsto w$ is a semiconjugacy of f with the rational function $\hat{f} = (1 - 2/w)^2$. By functoriality of the iterated monodromy groups (see [Nek08b]), the projection induces a group homomorphism

$$\nu : \text{IMG}(f) \rightarrow \text{IMG}(\hat{f})$$

and a morphism

$$\mu : \mathfrak{M}_f \rightarrow \mathfrak{M}_{\hat{f}}$$

of the corresponding self-similarity bimodules such that

$$\mu(g_1 \cdot x \cdot g_2) = \nu(g_1) \cdot \mu(x) \cdot \nu(g_2)$$

for all $g_1, g_2 \in \text{IMG}(f)$ and $x \in \mathfrak{M}_f$. The images of the generators α, β, γ in $\text{IMG}(\hat{f})$ are trivial (since they correspond to loops in which w is constant). The images of the generators s and t are generators of $\text{IMG}(\hat{f})$ (which we will also denote s and t) corresponding to the loops around the post-critical points 0 and 1 of \hat{f} , respectively (see Figure 2). The basis elements $1, 2 \in \mathbf{X}$ will be mapped by μ to the same element of $\mathfrak{M}_{\hat{f}}$, since the corresponding coset representatives ε, α are mapped to the same element ε of $\text{IMG}(\hat{f})$. Similarly the elements $3 \cdot b, 4 \cdot b \in \mathfrak{M}_f$ corresponding to the last two coordinates of the wreath recursion from Theorem 4.3 will be mapped to the same element by μ .

Consequently, applying the maps ν and μ to the wreath recursion of Theorem 4.3, we get the following wreath recursion generating $\text{IMG}(\hat{f})$

$$(10) \quad t = (t^{-1}s^{-1}, t), \quad s = \sigma,$$

where σ is the transposition. If $\{\mathbf{1}', \mathbf{2}'\}$ is the basis of the bimodule $\mathfrak{M}_{\hat{f}}$ corresponding to this wreath recursion, then

$$\mu(1) = \mu(2) = \mathbf{1}', \quad \mu(3 \cdot b) = \mu(4 \cdot b) = \mathbf{2}'.$$

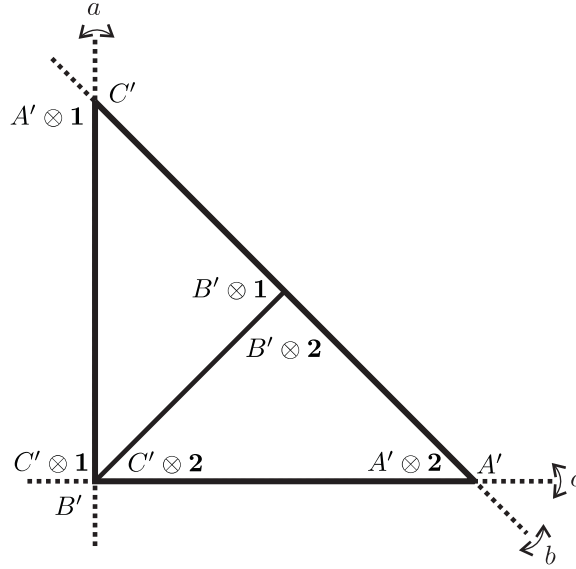
Let $\hat{\Gamma}$ be the group generated by

$$a = \sigma, \quad b = (a, c), \quad c = (b, b).$$

It is a quotient of the group Γ as a self-similar group. The corresponding epimorphisms of groups and self-similarity bimodules (which we will also denote by $\mu : \mathfrak{M} \rightarrow \mathfrak{M}_{\hat{\Gamma}}$ and $\nu : \Gamma \rightarrow \hat{\Gamma}$, as they are extensions of the maps μ and ν) are defined by

$$\mu(1) = \mu(2) = \mathbf{1}, \quad \mu(3) = \mu(4) = \mathbf{2},$$

and by $\nu(\alpha) = \varepsilon$, $\nu(\beta) = \varepsilon$, $\nu(\gamma) = \varepsilon$, $\nu(a) = a$, $\nu(b) = b$, $\nu(c) = c$, where $\mathbf{1}' = \mathbf{1}$ and $\mathbf{2}' = \mathbf{2} \cdot b$.

FIGURE 9. Triangle \mathcal{D}

It is proved in [Nek07] that the group $\widehat{\Gamma}$ is given by the presentation

$$\widehat{\Gamma} \cong \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (ab)^4 = (bc)^4 = \varepsilon \rangle,$$

and hence is isomorphic to the group generated by reflections of the Euclidean space with respect to the sides of an isosceles rectangular triangle (so that b is the reflection with respect to the hypotenuse).

Let us take, for instance, triangle $\mathcal{D} \subset \mathbb{R}^2$ with the vertices $A' = (1, 0)$, $B' = (0, 0)$, and $C' = (1, 1)$. Let the generators a , b , and c correspond to reflections with respect to the lines $B'C'$, $A'C'$, and $B'A'$, respectively, as it is shown on Figure 9.

It follows from Proposition 5.1 (and it is also easy to prove directly, see [Nek07]) that the nucleus of $\widehat{\Gamma}$ is the union of the groups

$$\begin{aligned} \widehat{\Gamma}_A &= \langle b, c \rangle = \nu(\Gamma_A) = \nu(\Gamma_{A_1}) \cong D_4, \\ \widehat{\Gamma}_B &= \langle a, c \rangle = \nu(\Gamma_B) = \nu(\Gamma_{B_1}) \cong D_2, \\ \widehat{\Gamma}_C &= \langle a, b \rangle = \nu(\Gamma_C) = \nu(\Gamma_{C_1}) \cong D_4. \end{aligned}$$

The pairwise intersections of the subgroups of the nucleus are the obvious ones:

$$\widehat{\Gamma}_{CB} = \langle a, b \rangle \cap \langle a, c \rangle = \langle a \rangle, \quad \widehat{\Gamma}_{CA} = \langle a, b \rangle \cap \langle b, c \rangle = \langle b \rangle, \quad \widehat{\Gamma}_{AB} = \langle b, c \rangle \cap \langle a, c \rangle = \langle c \rangle,$$

and are groups of order two. Denote

$$\widehat{\mathcal{G}} = \left\{ \widehat{\Gamma}_A, \widehat{\Gamma}_B, \widehat{\Gamma}_C, \widehat{\Gamma}_{BC}, \widehat{\Gamma}_{CA}, \widehat{\Gamma}_{AB}, \{\varepsilon\} \right\},$$

and let $\widehat{\Xi}$ be the image of the complex Ξ under the map $\nu : \Gamma \rightarrow \widehat{\Gamma}$, i.e., the complex associated with the poset of sets of the form $G \cdot h$ for $h \in \widehat{\Gamma}$ and $G \in \widehat{\mathcal{G}}$. The complex $\widehat{\Xi}$ is isomorphic to the barycentric subdivision of the simplicial complex obtained by tiling the Euclidean plane by the images of the triangle \mathcal{D} under the action of the group $\widehat{\Gamma}$. The groups $\widehat{\Gamma}_A, \widehat{\Gamma}_B, \widehat{\Gamma}_C$ correspond to the vertices A', B', C' of the

triangle \mathcal{D} , respectively. The groups $\widehat{\Gamma}_{AB}, \widehat{\Gamma}_{BC}, \widehat{\Gamma}_{AC}$ correspond to the edges $A'B', B'C', A'C'$, respectively. Each group is the stabilizer of the corresponding simplex of $\widehat{\Xi}$. The triangle \mathcal{D} is a fundamental domain of the action of $\widehat{\Gamma}$ on $\widehat{\Xi}$.

Define the map $P : \mathcal{T}_0 \rightarrow \mathcal{D}$ so that it is affine on the tetrahedra A_1ABC, B_1ABC, C_1ABC and acts on their vertices by the rules

$$(11) \quad P(A) = A', \quad P(B) = B', \quad P(C) = C',$$

$$(12) \quad P(A_1) = A', \quad P(B_1) = B', \quad P(C_1) = C',$$

i.e., mapping a vertex corresponding to $\Gamma_X \in \mathcal{G}$ to the vertex corresponding to its image under ν . Recall that we have introduced in Subsection 5.6 a Euclidean structure on \mathcal{T}_0 identifying it with a subset of \mathbb{R}^5 . The points A, B, C become vertices of an isosceles right triangle after this identification. The vectors $\overrightarrow{AA_1}, \overrightarrow{BB_1}$, and $\overrightarrow{CC_1}$ are orthogonal to the triangle ABC . Consequently, if we identify A', B', C' with the points A, B, C of \mathbb{R}^5 , then P will be the orthogonal projection of \mathcal{T}_0 onto the plain spanned by \overrightarrow{BA} and \overrightarrow{BC} .

The map $P : \mathcal{T}_0 \rightarrow \mathcal{D}$ can be extended to a continuous map $P : \Xi \rightarrow \widehat{\Xi}$ such that

$$(13) \quad P(\xi \cdot g) = P(\xi) \cdot \nu(g)$$

for every $\xi \in \Xi$ and $g \in \Gamma$. The map P will map the vertex corresponding to a coset $\Gamma_X \cdot g$ to the vertex corresponding to the coset $\nu(\Gamma_X \cdot g)$.

Denote by $L : \widehat{\Xi} \otimes \mathfrak{M}_{\widehat{\Gamma}} \rightarrow \widehat{\Xi}$ the $\widehat{\Gamma}$ -equivariant map induced by $I : \Xi \otimes \mathfrak{M} \rightarrow \Xi$, where $\mathfrak{M}_{\widehat{\Gamma}}$ is the self-similarity $\widehat{\Gamma}$ -bimodule. It is given by

$$L(P(\xi \otimes v)) = P(I(\xi \otimes v)),$$

and is well defined by equivariance of I and property (13).

Formulae defining I imply that

$$L(A' \otimes \mathbf{1}) = C', \quad L(B' \otimes \mathbf{1}) = (A'C'), \quad L(C' \otimes \mathbf{1}) = B',$$

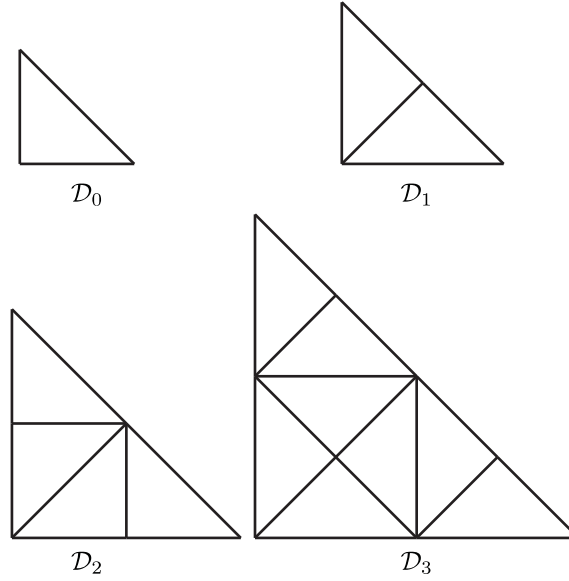
$$L(A' \otimes \mathbf{2}) = A', \quad L(B' \otimes \mathbf{2}) = (A'C'), \quad L(C' \otimes \mathbf{2}) = B'.$$

One checks directly using equivariance of the map L (see also [Nek08b, 4.8.3]) that the maps $\xi \mapsto L(\xi \otimes x)$ act on $\widehat{\Xi} \approx \mathbb{R}^2$ by affine transformations with the linear parts $\begin{pmatrix} -1/2 & -1/2 \\ 1/2 & -1/2 \end{pmatrix}$ and $\begin{pmatrix} 1/2 & -1/2 \\ -1/2 & -1/2 \end{pmatrix}$ for $x = \mathbf{1}$ and $x = \mathbf{2}$, respectively. Compare these matrices with the top left corners of the matrices \mathcal{I}_x in the proof of Proposition 5.5. Note that in both cases the transformation $L(\cdot \otimes x)$ divides all distances of \mathbb{R}^2 by $\sqrt{2}$.

It is proved in [Nek08b] that the map $L : \widehat{\Xi} \otimes \mathfrak{M}_{\widehat{\Gamma}} \rightarrow \widehat{\Xi}$ and hence the maps $L_n : \widehat{\Xi} \otimes \mathfrak{M}_{\widehat{\Gamma}}^{\otimes(n+1)} \rightarrow \widehat{\Xi} \otimes \mathfrak{M}_{\widehat{\Gamma}}^{\otimes n}$ are homeomorphisms.

By equivariance of the maps L_n , the action of $\widehat{\Gamma}$ on $\widehat{\Xi} \otimes \mathfrak{M}^{\otimes n}$ is obtained by conjugating the action of $\widehat{\Gamma}$ on $\widehat{\Xi}$ by the homeomorphism L_n . One can show that the fundamental domains $\mathcal{D}_n = \mathcal{D} \otimes \{\mathbf{1}, \mathbf{2}\}^n$ are rectangular isosceles triangles of area 2^{n-1} tiled by isometric copies of the triangle \mathcal{D} , as it is shown on Figure 6.1. The orbispaces $\widehat{\Xi} \otimes \mathfrak{M}^{\otimes n} / \widehat{\Gamma}$ can be identified with the fundamental domains \mathcal{D}_n (i.e., the natural map from \mathcal{D}_n to the orbispace is a homeomorphism).

The spaces $\mathcal{S}_n = \widehat{\Xi} \otimes \mathfrak{M}^{\otimes n} / \text{IMG}(\widehat{f})$ are obtained by “doubling” the triangles: by taking two copies \mathcal{D}_n and $\mathcal{D}_n \cdot a$, and pasting them together along the boundary.

FIGURE 10. The sets \mathcal{D}_n

Denote by

$$q_n : \mathcal{S}_{n+1} \rightarrow \mathcal{S}_n, \quad \lambda_n : \mathcal{S}_{n+1} \rightarrow \mathcal{S}_n$$

the covering induced by the projection $\zeta \otimes vx \mapsto \zeta \otimes v$ and the map induced by L_n . We will also denote $\mathcal{S} = \mathcal{S}_0$, $q = q_0$, and $\lambda = \lambda_0$. The maps λ_n act on each copy of \mathcal{D}_{n+1} in \mathcal{S}_{n+1} as similitudes with coefficient $\sqrt{2}/2$. The covering q_n maps halves of each copy of \mathcal{D}_{n+1} isometrically to copies of \mathcal{D}_n in \mathcal{S}_n (according to the rules similar to the rules of Proposition 5.9).

We can identify the spaces $\widehat{\Xi} \otimes \mathfrak{M}_{\widehat{\Gamma}}^{\otimes n}$ with \mathbb{C} in such a way that the group $\widehat{\Gamma}$ acts on them as the group of all isometries of the lattice $\mathbb{Z}[i]$, and the map L_n is identical (see [Nek08b, 4.8.2–3] for details). The subgroup $\text{IMG}(\widehat{f})$ of $\widehat{\Gamma}$ acts then on $\widehat{\Xi} \otimes \mathfrak{M}_{\widehat{\Gamma}}^{\otimes n} \approx \mathbb{C}$ as the group of affine transformations of the form $z \mapsto i^k z + z_0$, where $k \in \mathbb{Z}$ and $z_0 \in \mathbb{Z}[i]$. The coverings $q_n : \mathcal{S}_{n+1} \rightarrow \mathcal{S}_n$ are induced then by the map $z \mapsto (1-i)z$.

It follows that the limit dynamical system of $\text{IMG}(\widehat{f})$ is conjugate with the map $F_{i-1} : \mathcal{S}_0 \rightarrow \mathcal{S}_0$ induced by the transformation $z \mapsto (1-i)z$ of \mathbb{C} . It is well known (and follows from the general theory of iterated monodromy groups) that the map F_{1-i} is conjugate to the rational function \widehat{f} . The conjugating map is induced on \mathcal{S}_0 by the map $z \mapsto (\wp(z))^2$, where \wp is the Weierstrass function associated with the lattice $\mathbb{Z}[i]$.

6.2. Fibers of $\mathcal{M}_n \rightarrow \mathcal{S}_n$. Denote by $P_n : \Xi \otimes \mathfrak{M}^{\otimes n} \rightarrow \widehat{\Xi} \otimes \mathfrak{M}_{\widehat{\Gamma}}^{\otimes n}$ the map given by

$$P_n(\xi \otimes v) = P(\xi) \otimes \mu(v),$$

where $\xi \in \Xi$ and $v \in \mathfrak{M}^{\otimes n}$. We also denote $P_0 = P$.

The next proposition follows directly from the definitions.

Proposition 6.1. *The map $P_n : \Xi \otimes \mathfrak{M}^{\otimes(n+1)} \rightarrow \widehat{\Xi} \otimes \mathfrak{M}_F^{\otimes n}$ induces maps $\rho_n : \mathcal{M}_n \rightarrow \mathcal{S}_n$ making the diagrams*

$$\begin{array}{ccc} \mathcal{M}_{n+1} & \xrightarrow{p_n} & \mathcal{M}_n \\ \downarrow \rho_{n+1} & & \downarrow \rho_n \\ \mathcal{S}_{n+1} & \xrightarrow{q_n} & \mathcal{S}_n \end{array}, \quad \begin{array}{ccc} \mathcal{M}_{n+1} & \xrightarrow{\iota_n} & \mathcal{M}_n \\ \downarrow \rho_{n+1} & & \downarrow \rho_n \\ \mathcal{S}_{n+1} & \xrightarrow{\lambda_n} & \mathcal{S}_n \end{array}$$

commutative.

Denote by $Z_{\alpha,n}, Z_{\beta,n}, Z_{\gamma,n}$ the sets of points of \mathcal{M}_n of the form $[\xi]$ and $[\xi \cdot a]$, where $\xi \in \mathcal{T}_n$ is a fixed point of the transformation $\kappa_{\alpha,n}, \kappa_{\beta,n}, \kappa_{\gamma,n}$, respectively.

Theorem 6.2. *The set $\rho_n^{-1}(\xi)$ is a finite tree for every $\xi \in \mathcal{S}_n$.*

The map $p_n : \mathcal{M}_{n+1} \rightarrow \mathcal{M}_n$ restricts for every $\xi \in \mathcal{M}_{n+1}$ to a degree two branched covering $\rho_{n+1}^{-1}(\xi) \rightarrow \rho_n^{-1}(q_n(\xi))$. Denote by $Z_{n+1}(\xi)$ its critical point.

Intersections of $Z_{\alpha,n}, Z_{\beta,n}$, and $Z_{\gamma,n}$ with $\rho_n^{-1}(\xi)$ are singletons. Let us denote them by $Z_{\alpha,n}(\xi), Z_{\beta,n}(\xi)$, and $Z_{\gamma,n}(\xi)$, respectively.

Let $\xi = [\zeta \otimes x] \in \mathcal{S}_{n+1}$, where $\zeta \in \mathcal{D}_n$ and $x \in \{\mathbf{1}, \mathbf{1} \cdot a, \mathbf{2}, \mathbf{2} \cdot a\}$. The tree $\rho_{n+1}^{-1}(\xi)$ is union of two trees T_1 and T_2 such that

- (i) *the common point of T_1 and T_2 is $Z_{n+1}(\xi)$;*
- (ii) *restrictions of the map $p_n : \rho_{n+1}^{-1}(\xi) \rightarrow \rho_n^{-1}(q_n(\xi))$ onto T_1 and T_2 are homeomorphisms;*
- (iii) *$p_n(Z_{n+1}(\xi)) = Z_{\alpha,n}(q_n(\xi))$;*
- (iv) *$p_n^{-1}(Z_{\beta,n}(q_n(\xi))) \cap T_1 = Z_{\alpha,n+1}(\xi)$, and $p_n^{-1}(Z_{\beta,n}(q_n(\xi))) \cap T_2 = Z_{\gamma,n+1}(\xi)$;*
- (v) *if $x \in \{\mathbf{1}, \mathbf{1} \cdot a\}$, then $p_n^{-1}(Z_{\gamma,n}(q_n(\xi))) \cap T_1 = Z_{\beta,n+1}(\xi)$;*
- (vi) *if $x \in \{\mathbf{2}, \mathbf{2} \cdot a\}$, then $p_n^{-1}(Z_{\gamma,n}(q_n(\xi))) \cap T_2 = Z_{\beta,n+1}(\xi)$.*

Proof. The set $\rho_{n+1}^{-1}([\zeta \otimes x])$ is equal to the set of points of the form $[\xi \otimes y]$, where $\xi \in \mathcal{T}_n$ and $y \in \mathbf{X} \cdot \{\varepsilon, a\}$ are such that $P_n(\xi) = \zeta$ and $\mu(y) = x$. Note that for every $x \in \{\mathbf{1}, \mathbf{2}\}$ there exists exactly two elements $y_1, y_2 \in \mathbf{X} \cdot \{\varepsilon, a\}$ such that $\mu(y) = x$.

Depending on x , restriction of the map p_{n+1} onto $\rho_{n+1}^{-1}([\zeta \otimes x])$ acts by the rule

$$[\eta \otimes y] \mapsto [\eta], \quad \text{or} \quad [\eta \otimes y] \mapsto [\eta \cdot a],$$

as it is described in Proposition 5.9.

It follows that the restriction is two-to-one except for the points η such that $[\eta \otimes y_1] = [\eta \otimes y_2]$, where $\{y_1, y_2\} = \mu^{-1}(x)$. It is sufficient to consider the case $x \in \{\mathbf{1}, \mathbf{2}\}$ and $y \in \mathbf{X}$. The equality $[\eta \otimes y_1] = [\eta \otimes y_2]$ is equivalent to existence of an element $g \in \mathcal{N}$ such that $\eta \cdot g = \eta$ and $g \cdot y_1 = y_2 \cdot h$ for $h \in \text{IMG}(f)$. Note that since $\mu(y_1) = \mu(y_2) = x$, the element $g \in \mathcal{N}$ has an even number of factors a in its decomposition into a product of generators. It follows now from $h \in \text{IMG}(f)$ that g has an even total number of factors b and c , i.e., that g is an element of the set $\langle \alpha, \beta, \gamma \rangle \cdot \langle bc \rangle$. Since $y_1 \neq y_2$, the element g contains an odd number of factors α . Looking through the groups $\Gamma_A, \Gamma_B, \Gamma_C, \Gamma_{A_1}, \Gamma_{B_1}, \Gamma_{C_1}$, we conclude that

$$g \in \langle \alpha, \beta \rangle \cup \langle \alpha, \gamma \rangle \cup \alpha \cdot \langle bc \rangle.$$

Consequently, either $g = \alpha$, or $g \in \mathcal{N} \setminus \mathfrak{A}$ and the stabilizer of the point η contains α , by Lemma 5.3. In both cases $\eta \cdot \alpha = \eta$, i.e., $[\eta] \in Z_{\alpha,n}$. In the other direction, if $[\eta] \in Z_{\alpha,n}$, then $[\eta] = [\eta']$ for $\eta' \cdot \alpha = \eta'$, and then $[\eta' \otimes y_1] = [\eta' \cdot \alpha \otimes y_1] = [\eta' \otimes y_2]$.

We have shown that a point $[\eta]$ has one preimage under the restriction of ρ_{n+1} onto $\rho_{n+1}^{-1}([\zeta \otimes x])$ if and only if $[\eta] \in Z_{\alpha,n}$. In all the other cases it has two preimages.

Let us prove by induction on n that the intersections of $Z_{\alpha,n}$, $Z_{\beta,n}$, and $Z_{\gamma,n}$ with the fibers of the map ρ_n are singletons.

It is true for $n = 0$. By Theorem 5.4, a point $\eta \otimes y \in \mathcal{T}_{n+1}$, for $\eta \in \mathcal{T}_n$ and $y \in \mathbf{X}$, belongs to $K_{\alpha,n+1}$ if and only if $\eta \in K_{\beta,n}$ and $y \in \{1, 3\}$.

We have

$$\kappa_{\alpha,n+1}(\eta \otimes y) = \kappa_{\beta,n}(\eta) \otimes y = \eta \cdot \beta \otimes y.$$

If $\eta \otimes y = \eta \cdot \beta \otimes y$, then it follows from Theorem 5.4 that $\eta \cdot \beta = \eta \cdot g$ and $g \cdot y = y$ for $g \in \langle a, \alpha, \gamma \rangle$. But then $\eta = \eta \cdot \beta g^{-1}$, which implies that $\beta g^{-1} \in \mathcal{N}$, and then by Lemma 5.3 that η is fixed under β .

It follows now from the inductive assumption (since $\mu(1) \neq \mu(3)$) that intersection of $Z_{\alpha,n+1}$ with the fibers of ρ_{n+1} are singletons.

Similarly, $\eta \otimes y$ belongs to $K_{\gamma,n+1}$ if and only if $\eta \in K_{\beta,n}$ and $y \in \{2, 4\}$. By the same arguments as above, $\eta \otimes y$ is fixed under $\kappa_{\gamma,n+1}$ if and only if η is fixed under $\kappa_{\beta,n}$. Consequently, intersections of $Z_{\gamma,n+1}$ with the fibers of ρ_{n+1} are singletons.

A point $\eta \otimes y$ belongs to $K_{\beta,n+1}$ if and only if $\eta \in K_{\gamma,n}$ and $y \in \{1, 4\}$, or $\eta \in K_{a\alpha\gamma,n}$ and $y \in \{1, 4\}$.

Suppose that $\eta \in K_{\gamma,n}$. Then

$$\kappa_{\beta,n+1}(\eta \otimes y) = \eta \cdot \gamma \otimes y.$$

Again, by Theorem 5.4, if $\eta \otimes y = \eta \cdot \gamma \otimes y$, then $\eta \cdot \gamma = \eta \cdot g$ and $g \cdot y = y$ for some $g \in \langle \alpha, \gamma, a \rangle$. The equality $g(y) = y$ implies that $g \in \{\varepsilon, \gamma, \gamma^\alpha\}$ (as $\gamma^a = \gamma^\alpha$ and $\langle a, \alpha \rangle$ acts faithfully on \mathbf{X}). The case $g = \gamma$ contradicts $g \cdot y = y$ and $y \in \{1, 4\}$. If $g = \gamma^\alpha$, then η is fixed by $\gamma\gamma^\alpha$, which implies by Lemma 5.3 that it is also fixed by γ . If $g = \varepsilon$, then $\eta \cdot g = \eta \cdot \gamma$ implies that η is fixed by γ .

Suppose that $\eta \in K_{a\alpha\gamma,n}$ and $y \in \{1, 4\}$. Then $\kappa_{\beta,n+1}(\eta \otimes y) = \eta \cdot a\alpha\gamma \otimes \sigma\pi(y)$. If $\eta \otimes y = \eta \cdot a\alpha\gamma \otimes \sigma\pi(y)$, then there exists $g \in \langle \alpha, a, \gamma \rangle$ such that $\eta \cdot g = \eta \cdot a\alpha\gamma$ and $g \cdot y = \sigma\pi(y)$. The condition $g(y) = \sigma\pi(y)$ implies that $g \in \{a\alpha, a\alpha\gamma, a\alpha\gamma^\alpha\}$. The condition $g|_y = \varepsilon$ eliminates the case $g = a\alpha\gamma$. If $g = a\alpha$, then $\eta \cdot g = \eta \cdot a\alpha\gamma$ implies $\eta \cdot \gamma = \xi$ (since $a\alpha$ commutes with γ). If $g = a\alpha\gamma^\alpha$, then $\eta = \eta \cdot \gamma^\alpha$, which also by Lemma 5.3 implies that $\eta = \eta \cdot \gamma$.

We see that in all cases $[\eta] \in Z_{\gamma,n}$, hence intersections of $Z_{\beta,n+1}$ with the fibers of ρ_{n+1} are singletons.

Note that we have already proved that

$$p_n(Z_{\alpha,n+1}(\xi)) = p_n(Z_{\gamma,n+1}(\xi)) = Z_{\beta,n}(q_n(\xi))$$

and $p_n(Z_{\beta,n+1}(\xi)) = Z_{\gamma,n}(q_n(\xi))$. The equality $p_n(Z_{n+1}(\xi)) = Z_{\alpha,n}(q_n(\xi))$ was also proved before.

By the proved above, the fiber $\rho_{n+1}^{-1}(\zeta \otimes x)$ is obtained by identifying in the sets $\rho_n^{-1}(\zeta) \otimes y_1$ and $\rho_n^{-1}(\zeta) \otimes y_2$ the points $Z_{\alpha,n}(\zeta) \otimes y_1$ with $Z_{\alpha,n}(\zeta) \otimes y_2$. Since the fibers of ρ_0 are trees, we get by induction that the fibers of ρ_n are also trees. It also follows that the point $Z_{n+1}(\zeta \otimes x) = Z_{\alpha,n}(\zeta) \otimes y_1 = Z_{\alpha,n}(\zeta) \otimes y_2$ separates the subtrees $\rho_n^{-1}(\zeta) \otimes y_1$ and $\rho_n^{-1}(\zeta) \otimes y_2$. Let $y_1 \in \{1, 1 \cdot a, 3, 3 \cdot a\}$ and $y_2 \in \{2, 2 \cdot a, 4, 4 \cdot a\}$. Then denote $T_1 = \rho_n^{-1}(\zeta) \otimes y_1$ and $T_2 = \rho_n^{-1}(\zeta) \otimes y_2$.

The rest of the theorem follows now from the recurrent description of the sets $Z_{\alpha,n}$, $Z_{\beta,n}$, and $Z_{\gamma,n}$, and the sets $K_{\alpha,n}$, $K_{\beta,n}$, and $K_{\gamma,n}$. \square

6.3. Unfolding trees and the map ι_n . Consider the embedding of the complex \mathcal{T}_0 into \mathbb{R}^5 described in Subsection 5.6 and the embedding of \mathcal{D} into \mathbb{R}^2 described in Subsection 6.1. Then projection $P : \mathcal{T}_0 \rightarrow \mathcal{D}$ acts as the map $(x_1, x_2, x_3, x_4, x_5) \mapsto$

(x_1, x_2) . The fiber $P^{-1}(x_1, x_2)$ is a union of three segments with one common end $(x_1, x_2, 0, 0, 0)$. The other ends of the segments are $Z_\alpha = Z_{\alpha,0}((x_1, x_2))$, $Z_\beta = Z_{\beta,0}((x_1, x_2))$, $Z_\gamma = Z_{\gamma,0}((x_1, x_2))$, respectively.

The point Z_α is the intersection of $P^{-1}(x_1, x_2)$ with the plane A_1BC , hence it has coordinates $(x_1, x_2, x_1, 0, 0)$. The coordinates of the point

$$Z_\beta = P^{-1}(x_1, x_2) \cap B_1AC$$

are $(x_1, x_2, 0, 1 - x_1 - x_2, 0)$. The coordinates of the point

$$Z_\gamma = P^{-1}(x_1, x_2) \cap C_1AB$$

are $(x_1, x_2, 0, 0, x_2)$.

Consequently, $P^{-1}(x_1, x_2)$ is a tripod with feet Z_α , Z_β , and Z_γ and lengths of legs x_1 , $1 - x_1 - x_2$, and x_2 , respectively.

We see that the set of the graphs $P^{-1}(\xi)$ for $\xi \in \mathcal{D}$ coincides with the set of all tripods with feet marked by $Z_\alpha, Z_\beta, Z_\gamma$ such that the sum of lengths of legs (the *mass* of the tripod) is equal to one. The triangle \mathcal{D} is interpreted as the configuration space of such tripods.

Theorem 6.2 is reformulated then as the following recurrent procedure of constructing the fibers $\rho_n^{-1}(\xi)$.

A *marked tree* is a finite tree together with a marking of three points by the letters Z_α , Z_β , and Z_γ . We assume that not all marked points coincide.

Proposition 6.3. *Let T be a marked tree. Take two copies T_1 and T_2 of T and paste them together, identifying the copies of the point marked by Z_α . Mark in the obtained tree $\Phi_i(T)$ (for $i = 1, 2$) the copy of the point Z_β in T_1 by Z_α , the copy of Z_β in T_2 by Z_γ . If $i = 1$, then mark the copy of Z_γ in T_1 by Z_β . Otherwise, mark by Z_β the copy of Z_γ in T_2 .*

Then for every $\xi \in \mathcal{D}$, $x_1 x_2 \dots x_n \in \{1, 2\}^n$, and $\delta \in \{\varepsilon, a\}$ the tree $\rho_n^{-1}(\xi \otimes x_1 x_2 \dots x_n \cdot \delta)$ is isomorphic as a marked tree to

$$\Phi_{x_n} \circ \dots \circ \Phi_{x_2} \circ \Phi_{x_1}(\rho^{-1}(\xi)).$$

The covering $p_n : \rho_{n+1}^{-1}(\xi \otimes x_1 x_2 \dots x_{n+1}) \rightarrow \rho_n^{-1}(\xi \otimes x_1 x_2 \dots x_n)$ maps the copies T_1 and T_2 of $\rho_n^{-1}(\xi \otimes x_1 x_2 \dots x_n)$ tautologically onto $\rho_n^{-1}(\xi \otimes x_1 x_2 \dots x_n)$.

See Figure 11 for a description of the maps Φ_1 and Φ_2 . We denote by $Z'_\alpha, Z'_\beta, Z'_\gamma$ the copies of $Z_\alpha, Z_\beta, Z_\gamma$ in T_1 , and by $Z''_\alpha, Z''_\beta, Z''_\gamma$ the copies of $Z_\alpha, Z_\beta, Z_\gamma$ in T_2 .

If T is a marked tree, then we call the tripod $Z_\alpha Z_\beta Z_\gamma$ (i.e., the convex hull of the points Z_α, Z_β , and Z_γ) the *Hubbard tripod* of T . Let us denote by O the common point of the legs of the Hubbard tripod.

Let us describe how the map $\iota : \mathcal{M}_1 \rightarrow \mathcal{M}$ acts on the fibers of the maps ρ_1 and ρ . Denote by $x^{(1)}$ the element $1, 1 \cdot a, 4, 4 \cdot a$, if $x = 1, 1 \cdot a, 2, 2 \cdot a$, respectively. Denote $x^{(2)} = 2, 2 \cdot a, 3, 3 \cdot a$, if $x = 1, 1 \cdot a, 2, 2 \cdot a$, respectively. Then $\{x^{(1)}, x^{(2)}\} = \mu^{-1}(x)$, so that $\rho_1^{-1}(\xi \otimes x) = \rho^{-1}(\xi) \otimes x^{(1)} \cup \rho^{-1}(\xi) \otimes x^{(2)}$.

Proposition 6.4. *Let $\xi, \zeta \in \mathcal{S}$ and $x \in \{1, 1 \cdot a, 2, 2 \cdot a\}$ be such that $\lambda(\xi \otimes x) = \zeta$. Then $\iota : \rho_1^{-1}(\xi \otimes x) \rightarrow \rho^{-1}(\zeta)$ maps the edge connecting $Z_\gamma(\xi) \otimes x^{(1)}$ and $O(\xi) \otimes x^{(1)}$ isometrically to $Z_\beta(\zeta)O(\zeta)$; collapses the edge connecting $Z_\gamma(\xi) \otimes x^{(2)}$ and $O(\xi) \otimes x^{(2)}$ to one point; and maps the path connecting $Z_\beta(\xi) \otimes x^{(1)}$ and $Z_\beta(\xi) \otimes x^{(2)}$ to the path connecting $Z_\alpha(\zeta)$ and $Z_\beta(\zeta)$ dividing all the distances in it by two.*

If $x \in \{1, 1 \cdot a\}$, then $\iota(Z_\beta(\xi) \otimes x^{(1)}) = Z_\alpha(\zeta)$ and $\iota(Z_\beta(\xi) \otimes x^{(2)}) = Z_\gamma(\zeta)$, otherwise $\iota(Z_\beta(\xi) \otimes x^{(1)}) = Z_\gamma(\zeta)$ and $\iota(Z_\beta(\xi) \otimes x^{(2)}) = Z_\alpha(\zeta)$.

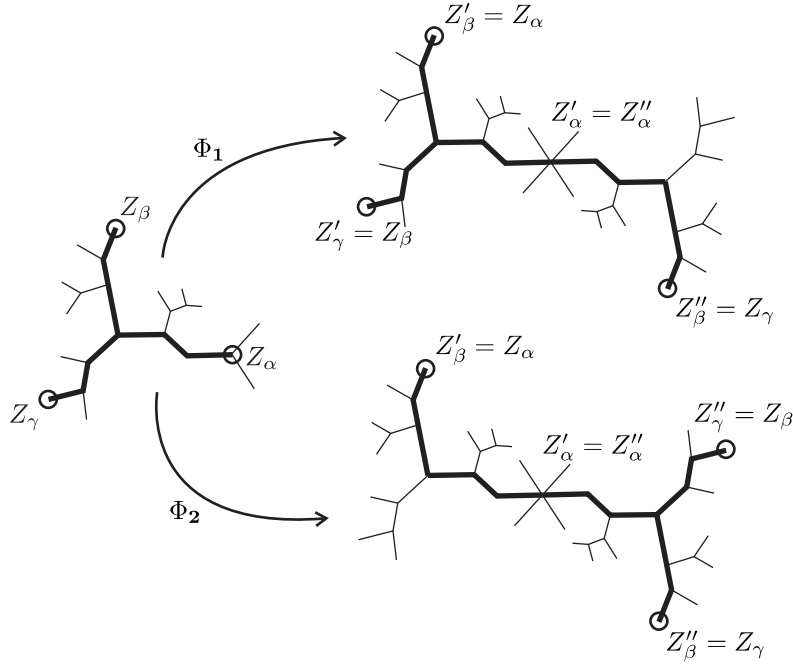


FIGURE 11. Unfolding transformations

Here we wrote Z_α , Z_β , and Z_γ instead of $Z_{\alpha,0}$, $Z_{\beta,0}$, and $Z_{\gamma,0}$.

Proof. Let $\xi = (x_1, x_2) \in \mathcal{D}$ (the case $\xi \in \mathcal{D} \cdot a$ is similar). Then $\rho^{-1}(\xi)$ is the tripod with lengths of legs $(x_1, x_2, 1 - x_1 - x_2)$.

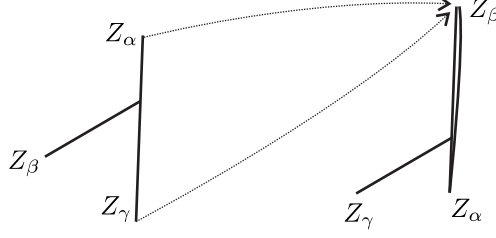
The map ι acts on $\rho^{-1}(\xi \otimes \mathbf{1})$ and on $\rho^{-1}(\xi \otimes \mathbf{1} \cdot a)$ by the rule (see the proof of Proposition 5.5):

$$\iota(Z_\alpha \otimes 1) = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathcal{I}_1 \begin{pmatrix} x_1 \\ x_2 \\ x_1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} (1 - x_1 - x_2)/2 \\ (1 + x_1 - x_2)/2 \\ 0 \\ 0 \\ x_1/2 \end{pmatrix},$$

$$\iota(Z_\beta \otimes 1) = \begin{pmatrix} (1 - x_1 - x_2)/2 \\ (1 + x_1 - x_2)/2 \\ (1 - x_1 - x_2)/2 \\ 0 \\ 0 \end{pmatrix} = Z_\alpha(\zeta),$$

similarly

$$\iota(Z_\gamma \otimes 1) = \begin{pmatrix} (1 - x_1 - x_2)/2 \\ (1 + x_1 - x_2)/2 \\ 0 \\ x_2 \\ 0 \end{pmatrix} = Z_\beta(\zeta),$$

FIGURE 12. The folding map F

$$\iota(O \otimes 1) = \begin{pmatrix} (1 - x_1 - x_2)/2 \\ (1 + x_1 - x_2)/2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = O(\zeta),$$

and

$$\begin{aligned} \iota(Z_\alpha \otimes 2) &= \iota(Z_\alpha \otimes 1), \\ \iota(Z_\beta \otimes 2) &= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mathcal{I}_2 \begin{pmatrix} x_1 \\ x_2 \\ 0 \\ 1 - x_1 - x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} (1 - x_1 - x_2)/2 \\ (1 + x_1 - x_2)/2 \\ 0 \\ 0 \\ (1 + x_1 - x_2)/2 \end{pmatrix} = Z_\gamma(\zeta) \\ \iota(Z_\gamma \otimes 2) &= \begin{pmatrix} (1 - x_1 - x_2)/2 \\ (1 + x_1 - x_2)/2 \\ 0 \\ 0 \\ x_1 \end{pmatrix} = \iota(O \otimes 2). \end{aligned}$$

The statement of the proposition follows now from the above formulae. The case $x = \mathbf{2}$ is analogical. \square

Corollary 6.5. *The map $\theta : \mathcal{M} \rightarrow \mathcal{M}_1$ is isometric on the edges $O(\zeta)Z_\beta(\zeta)$ of the trees $\rho^{-1}(\zeta)$ and multiplies the lengths of the legs $O(\zeta)Z_\alpha(\zeta)$ and $O(\zeta)Z_\gamma(\zeta)$ by two.*

6.4. Folding tripods and fibers of $(z, w) \mapsto w$. Consider the map $p_0 \circ \theta : \mathcal{M} \rightarrow \mathcal{M}$. Note that since $\lambda : \mathcal{S}_1 \rightarrow \mathcal{S}$ is a homeomorphism, we have $\rho(\theta(\xi)) = \lambda^{-1}(\rho(\xi))$ for all $\xi \in \mathcal{M}$.

It follows from the description of the fiberwise action of p_0 and θ that restriction $p_0 \circ \theta : \rho^{-1}(\xi) \rightarrow \rho^{-1}(q_0 \circ \lambda^{-1}(\xi))$ acts in the following way.

Double all the distances inside the legs $Z_\alpha(\xi)O(\xi)$ and $Z_\gamma(\xi)O(\xi)$ of $\rho^{-1}(\xi)$, and then fold the path $Z_\alpha(\xi)Z_\gamma(\xi)$ in two. The common image of $Z_\alpha(\xi)$ and $Z_\gamma(\xi)$ is $Z_\beta(\zeta)$, the image of the middle of the path $Z_\alpha(\xi)Z_\gamma(\xi)$ is $Z_\alpha(\zeta)$ and the image of the vertex $Z_\beta(\xi)$ is $Z_\gamma(\zeta)$ (see Figure 12).

If (x_1, x_2, x_3) are the lengths of the legs $Z_\alpha O$, $Z_\beta O$, and $Z_\gamma O$ in $\rho^{-1}(\xi)$, respectively, then the lengths of the corresponding legs in the tripod $\rho^{-1}(p_0 \circ \theta(\zeta))$ are

$$F(x_1, x_2, x_3) = (|x_1 - x_3|, \quad 2 \min(x_1, x_3), \quad x_2),$$

respectively. Recall that $x_1 + x_2 + x_3 = 1$ and note that $|x_1 - x_3| + 2 \min(x_1, x_3) + x_2 = x_1 + x_2 + x_3$.

Itinerary of a triple (x_1, x_2, x_3) is the sequences (k_1, k_2, \dots) , where $k_n = \mathbf{1}$ if in $F^{n-1}(x_1, x_2, x_3) = (y_1, y_2, y_3)$ we have $y_3 > y_1$; otherwise, $k_n = \mathbf{2}$.

Denote by $\rho_\infty : \mathcal{M}_\infty \rightarrow \mathcal{S}_\infty$ the limit of the maps $\rho_n : \mathcal{M}_n \rightarrow \mathcal{S}_n$, where $\mathcal{M}_\infty \approx \mathcal{J}_{\text{IMG}(f)}$ and $\mathcal{S}_\infty \approx \mathcal{J}_{\text{IMG}(\hat{f})}$ are the projective limits of \mathcal{M}_n and \mathcal{S}_n (with respect to the maps ι_n and λ_n), respectively. We denote $q^n = q_0 \circ q_1 \circ \dots \circ q_{n-1}$ and $\lambda^n = \lambda_0 \circ \lambda_1 \circ \dots \circ \lambda_{n-1}$. We have $q_n \circ \lambda_{n+1} = \lambda_n \circ q_{n+1}$ for all $n \geq 0$.

Proposition 6.6. *Let $(\zeta_0, \zeta_1 = \lambda^{-1}(\zeta_0), \zeta_2 = \lambda_1^{-1}(\zeta_1), \dots)$ be a point of \mathcal{S}_∞ and suppose that $\rho^{-1}(\zeta_0)$ is a tripod with the lengths of legs (x_1, x_2, x_3) . Let (k_1, k_2, \dots) be the itinerary of the tripod $\rho^{-1}(\zeta_0)$. Then the trees $\rho_n^{-1}(\zeta_n)$ are isometric to $\Phi_{k_1} \circ \dots \circ \Phi_{k_n}(T_n)$, where T_n is the tripod with lengths of legs $F^n(x_1, x_2, x_3)$.*

The maps $\iota_n : \rho_{n+1}^{-1}(\zeta_{n+1}) \rightarrow \rho_n^{-1}(\zeta_n)$ project each of the 2^n copies of the graph $\Phi_{k_{n+1}}(T_{n+1})$ in $\Phi_{k_1} \circ \dots \circ \Phi_{k_n}(\Phi_{k_{n+1}}(T_{n+1}))$ onto its Hubbard tripod, divides the lengths of the legs $Z_\alpha O$ and $Z_\gamma O$ by two, and then identifies each copy with the corresponding copy of T_n in $\Phi_{k_1} \circ \dots \circ \Phi_{k_n}(T_n)$.

Proof. If (k_1, k_2, \dots) is the itinerary of the tripod $\rho^{-1}(\zeta)$, then (k_2, k_3, \dots) is the itinerary of the tripod $\rho^{-1}(q_0 \circ \lambda_0^{-1}(\zeta)) = \rho^{-1}(q_0(\zeta_1))$, and $\zeta_1 \in \mathcal{D} \otimes k_1 \cdot \delta$ for $\delta \in \{\varepsilon, a\}$ (see the description of the action of $p_0 \circ \theta : \rho^{-1}(\zeta) \rightarrow \rho^{-1}(\lambda_0^{-1}(\zeta))$ and Proposition 6.3).

Note that

$$\lambda_0^{-1} \circ (q_0 \circ q_1 \circ \dots \circ q_{n-1}) = (q_1 \circ q_2 \circ \dots \circ q_n) \circ \lambda_n^{-1},$$

which implies

$$q_0 \circ \lambda_0^{-1} \circ q^n \circ (\lambda^n)^{-1} = q_0 \circ (q_1 \circ q_2 \circ \dots \circ q_n) \circ \lambda_n^{-1} \circ (\lambda^n)^{-1} = q^{n+1} \circ (\lambda^{n+1})^{-1}.$$

It follows now by induction that $(k_{n+1}, k_{n+2}, \dots)$ is the itinerary of the tripod $\rho^{-1}(q^n \circ (\lambda^n)^{-1}(\zeta)) = \rho^{-1}(q^n(\zeta_n))$, which has lengths of legs $F^n(x_1, x_2, x_3)$, and that $q_1 \circ \dots \circ q_{n-1}(\zeta_n) \in \mathcal{D} \otimes k_n \cdot \delta$ for some $\delta \in \{\varepsilon, a\}$.

We know that

$$\lambda_k(\mathcal{D}_k \otimes x) = \mathcal{D}_{k-1} \otimes x$$

for all $k \geq 1$ and $x \in \{\mathbf{1}, \mathbf{1} \cdot a, \mathbf{2}, \mathbf{2} \cdot a\}$.

Therefore, if $\zeta_1 \in \mathcal{D} \otimes k_1 \cdot \delta_1$ for $\delta_1 \in \{\varepsilon, a\}$, then

$$\zeta_n = (\lambda_1 \circ \dots \circ \lambda_{n-1})^{-1}(\zeta_1) \in \mathcal{D}_{n-1} \otimes k_1 \cdot \delta_1$$

for all n .

Similarly, if $\delta_m \in \{\varepsilon, a\}$ is such that $q_1 \circ \dots \circ q_{m-1}(\zeta_m) \in \mathcal{D} \otimes x_m \cdot \delta_m$, then for all $n > m$ we have

$$\begin{aligned} q_{n-m+1} \circ \dots \circ q_{n-1}(\zeta_n) &= \\ q_{n-m+1} \circ \dots \circ q_{n-1} \circ \lambda_{n-1}^{-1} \circ \dots \circ \lambda_m^{-1}(\zeta_m) &= \\ \lambda_{n-m}^{-1} \circ \dots \circ \lambda_1^{-1} \circ q_1 \circ \dots \circ q_{m-1}(\zeta_m) &\in \mathcal{D}_{n-m} \otimes x_m \cdot \delta_m, \end{aligned}$$

since $q_{i+1} \circ \lambda_{i+1}^{-1} = \lambda_i^{-1} \circ q_i$ for all i .

It follows that $\zeta_n \in \mathcal{D} \otimes k_n k_{n-1} \dots k_1 \cdot \delta_1$. The first paragraph of the proposition follows now from Proposition 6.3. The second paragraph follows from the definition of the map ι_n and Proposition 6.4. \square

6.5. The spaces \mathcal{M}_n as subsets of the Julia set. A *dendrite* is a path connected and locally path connected space without simple closed curves.

Proposition 6.7. *The fibers of the map $\rho_\infty : \mathcal{J}_{\text{IMG}(f)} \rightarrow \mathcal{J}_{\text{IMG}(\widehat{f})}$ are dendrites.*

Proof. By Proposition 6.6, every fiber of ρ_∞ is homeomorphic to the inverse limit A_∞ of a sequence of finite trees A_n with respect to maps $\iota_n : A_{n+1} \rightarrow A_n$, where ι_n is composition of a projection of A_{n+1} onto a subtree A'_{n+1} and a homeomorphism $A'_{n+1} \rightarrow A_n$.

Let us show that A_∞ is path connected. Let $x, y \in A_\infty$ be arbitrary points, and let x_n, y_n be their images in A_n . Let γ_n be the unique arc connecting x_n to y_n in A_n . The projection of A_n onto A'_n maps γ_n to its sub-arc γ'_n by mapping the connected components of $\gamma_n \setminus \gamma'_n$ to the endpoints of γ'_n . It follows that the inverse limit of the arcs γ_n is an arc in A_∞ connecting x to y .

Every open neighborhood of $x \in A_\infty$ contains a neighborhood which is the inverse image of an open subset of A_n for some n , hence it contains an open neighborhood which is the inverse limit of subtrees of the trees A_n . By the argument above, this inverse limit is path connected. Consequently, A_∞ is locally path connected.

Suppose that γ is an arc with endpoints $x, y \in A_\infty$. Denote by $\gamma_{(n)}$ its image in A_n , and let x_n, y_n be the images of x, y in A_n . The set $\gamma_{(n)}$ is connected, hence it has to contain the unique arc γ_n connecting x_n and y_n . It follows that γ contains the inverse limit of the arcs γ_n . But we have proved that the inverse limit of γ_n is an arc connecting x and y . Consequently, there exists only one arc connecting x and y in A_∞ . \square

Recall that $\widetilde{\theta}_n : \mathcal{M}_n \rightarrow \mathcal{M}_\infty$ denotes the limit of $\theta_{n+k-1} \circ \cdots \circ \theta_n : \mathcal{M}_n \rightarrow \mathcal{M}_{n+k}$ and is a homeomorphism of \mathcal{M}_n with $\widetilde{\theta}_n(\mathcal{M}_n)$. The spaces \mathcal{M}_n can be hence identified with subsets of the Julia set of f . Let us denote by $\widetilde{\mathcal{M}}_n$ the image of $\widetilde{\theta}_n(\mathcal{M}_n)$ under the natural homeomorphism of $\mathcal{M}_\infty \approx \mathcal{J}_{\text{IMG}(f)}$ with the Julia set of f .

Denote by $\widetilde{\iota}_n : \widetilde{\mathcal{M}}_{n+1} \rightarrow \widetilde{\mathcal{M}}_n$ the map obtained from ι_n after identification of the spaces \mathcal{M}_n with the sets $\widetilde{\mathcal{M}}_n$.

Proposition 6.8. *Intersections \mathcal{J}_{w_0} of the Julia set of f with the lines $w = w_0$ are dendrites. The set $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_0$ is the union of the convex hulls of the sets $\{(0, w_0), (1, w_0), (w_0, w_0)\}$ inside the dendrite \mathcal{J}_{w_0} .*

We have $\widetilde{\mathcal{M}}_n = f^{-n}(\widetilde{\mathcal{M}})$. The map $\widetilde{\iota}_n : \widetilde{\mathcal{M}}_{n+1} \rightarrow \widetilde{\mathcal{M}}_n$ acts as projection of the trees $\widetilde{\mathcal{M}}_{n+1} \cap \mathcal{J}_{w_0}$ onto their sub-trees $\widetilde{\mathcal{M}}_n \cap \mathcal{J}_{w_0}$.

If T is a tree and $T' \subset T$ is a sub-tree, then projection of T onto T' maps a point $t \in T$ to the end of the unique path γ starting at t , ending in a point t' of T' and such that the only common point of γ with T' is t' .

Proof. The fact that intersections of the Julia set of f with the lines $w = w_0$ are dendrites and the last paragraph of the theorem follow directly from Proposition 6.4 and Theorem 5.10.

By Theorem 6.2, for every $\xi \in \mathcal{S}_{n+1}$ the point $Z_{\alpha,n}(q_n(\xi))$ is the image of the critical point of the restriction of the covering p_n onto the fiber $\rho_{n+1}^{-1}(\xi)$. The critical point of the restriction $(1 - 2z/w_0)^2$ of f onto the fiber $w = w_0$ is $w_0/2$, and its image is 0. Consequently, the points Z_α correspond to the points $z = 0$ of the respective slices of the Julia set of f . Since $p_n(Z_{\alpha,n+1}(\xi)) = Z_{\beta,n}(q_n(\xi))$,

the points corresponding to Z_β are $z = 1$; since $p_n(Z_{\beta,n+1}(\xi)) = Z_{\gamma,n}(q_n(\xi))$, the points corresponding to Z_γ in the slice $w = w_0$ is the point $z = w_0$.

Since the maps θ_n are homeomorphic embeddings of trees, the set $\theta_{n-1} \circ \dots \circ \theta_0(\mathcal{M}) \subset \mathcal{M}_n$ is the union of the convex hulls of the points $Z_\alpha(\xi), Z_\beta(\xi), Z_\gamma(\xi)$ inside the fibers $\rho_n^{-1}(\xi)$. It follows that $\widetilde{\mathcal{M}}$ is the union of the hulls of the points $(0, w_0), (1, w_0), (w_0, w_0)$ inside the intersections of the Julia set with the lines $w = w_0$, which are dendrites by Theorem 6.7. \square

Note that the set $\rho_0^{-1}(\mathcal{D} \cap \mathcal{D} \cdot a) \subset \mathcal{M}$ is the union of tripods in which at least one leg has length zero. The map $(z, w) \mapsto (\bar{z}, \bar{w})$ is an automorphism of the dynamical system $(\mathbb{P}\mathbb{C}^2, f)$ and it changes the orientation of the Hubbard tripod of $J_{w_0} \subset \mathbb{C}$ to the opposite one (here the Hubbard tripod of J_{w_0} is the convex hull of the points $\{0, 1, w_0\}$ inside the dendrite J_{w_0} , i.e., intersection of J_{w_0} with \mathcal{M}). It follows that the Hubbard tripods of J_{w_0} for real values of w_0 have only one orientation, i.e., that they have at least one leg of length zero. The real line of the Riemann sphere is homeomorphic to a circle. The set $\mathcal{D} \cap \mathcal{D} \cdot a$ is also homeomorphic to a circle (it is the boundary of the triangle \mathcal{D}). It follows that the subset of the Julia set of f corresponding to $\rho_0^{-1}(\mathcal{D} \cap \mathcal{D} \cdot a)$ is precisely the union of the sets J_{w_0} for $w_0 \in \mathbb{R} \cup \{\infty\}$.

The virtual endomorphism ϕ from Proposition 4.2 corresponds to the letter $1 \in \mathbf{X}$ and is computed using the fixed point $(z, w) = (0.3002\dots + 0.3752\dots i, 2i)$. The Hubbard tripod $Z_\alpha Z_\beta Z_\gamma$ of the slice J_{2i} is oriented counterclockwise (see Figure 2).

It follows that the sets $\mathcal{T}_0 = \rho_0^{-1}(\mathcal{D}_0)$ and $\mathcal{T}_0 \cdot a = \rho_0^{-1}(\mathcal{D}_0 \cdot a)$ correspond to intersections of the set $\widetilde{\mathcal{M}}$ with the half-spaces $\Im(w) \geq 0$ and $\Im(w) \leq 0$, respectively. The Hubbard tripod of J_{w_0} is oriented counterclockwise if $\Im(w_0) > 0$ and clockwise if $\Im(w_0) < 0$.

7. MISCELLANY

7.1. A metric on the fibers of the Julia set. The folding transformation F used in Proposition 6.6 stretches the legs of tripods in a non-uniform way: two are stretched twice, one is not stretched at all. A modified transformation might be more natural. Let us show that it is essentially equivalent to the transformation F and use this fact to construct a metric on the dendrite slices of the Julia set of f .

Let T be a tripod with feet labeled by Z_α, Z_β , and Z_γ . Denote by $F_1(T)$ the tripod obtained by folding the path $Z_\alpha Z_\gamma$ in two and labeling the common image of Z_α and Z_β by Z_β , the image of Z_β by Z_γ , and the image of the midpoint of $Z_\alpha Z_\gamma$ by Z_α .

If (x, y, z) are lengths of the legs $Z_\alpha O, Z_\beta O$, and $Z_\gamma O$, respectively, then the corresponding lengths of legs of $F_1(T)$ are

$$\left(\frac{|x - z|}{2}, \min(x, z), y \right).$$

Definition 7.1. Denote $k(T) = \mathbf{1}$ if $x \leq z$, and $k(T) = \mathbf{2}$ if $x > z$. Then the F_1 -itinerary of a tripod T is the sequence k_0, k_1, \dots , where $k_n = k(F_1^n(T))$.

The fibers of the map $\rho : \mathcal{M} \rightarrow \mathcal{S}$ are *normalized* tripods, i.e., tripods of mass (sum of lengths of the legs) equal to one. Consider therefore the transformation \tilde{F} of tripods equal to F_1 followed by division of all distances in $F_1(T)$ by the mass of $F_1(T)$.

We will also denote by F_1 and \tilde{F} the corresponding transformations of triples of lengths of legs. Let, as before, F be the action of the map $p_0 \circ \theta_0$ on the fibers of ρ_0 , also seen as a map on the triples of lengths. We have

$$F(x, y, z) = (|x - z|, \quad 2 \min(x, z), \quad y).$$

The map \tilde{F} is given by

$$(14) \quad \tilde{F}(x, y, z) = \left(\frac{|x - z|}{1 + y}, \quad \frac{2 \min(x, z)}{1 + y}, \quad \frac{2y}{1 + y} \right).$$

Lemma 7.1. *The second iteration of \tilde{F} is uniformly expanding on every tripod by a factor not less than 2.*

Proof. The second iteration of \tilde{F} is equal to F_1^2 followed by dividing all distances by the mass of the obtained tripod. The lengths of the legs after one folding are either $((z - x)/2, x, y)$, or $((x - z)/2, z, y)$. Consequently, the triple of lengths of the legs of the tripod after two folding belongs to the list

$$\begin{aligned} \left(\frac{y - \frac{z-x}{2}}{2}, \frac{z-x}{2}, x \right) &= \left(\frac{x+2y-z}{4}, \frac{z-x}{2}, x \right), \\ \left(\frac{y - \frac{x-z}{2}}{2}, \frac{x-z}{2}, z \right) &= \left(\frac{-x+2y+z}{4}, \frac{x-z}{2}, z \right) \\ \left(\frac{\frac{z-x}{2} - y}{2}, y, x \right) &= \left(\frac{-x-2y+z}{4}, y, x \right), \\ \left(\frac{\frac{x-z}{2} - y}{2}, y, z \right) &= \left(\frac{x-2y-z}{4}, y, z \right). \end{aligned}$$

The mass of the folded tripod is $\frac{3x+2y+z}{4}$ if $x \leq z$, and $\frac{x+2y+3z}{4}$ if $x \geq z$. In each case the number is not more than $\frac{2x+2y+2z}{4} = 1/2$. \square

The branches of \tilde{F}^{-1} are the functions $\tilde{\Phi}_1, \tilde{\Phi}_2$ acting on the triples of lengths of legs by

$$(15) \quad \tilde{\Phi}_1(x, y, z) = \left(\frac{y}{1+x+y}, \quad \frac{z}{1+x+y}, \quad \frac{2x+y}{1+x+y} \right)$$

and

$$(16) \quad \tilde{\Phi}_2(x, y, z) = \left(\frac{2x+y}{1+x+y}, \quad \frac{z}{1+x+y}, \quad \frac{y}{1+x+y} \right).$$

Proposition 7.2. *The maps F and \tilde{F} acting on the triangle $\Delta = \{(x, y, z) : 0 \leq x, 0 \leq y, 0 \leq z, x + y + z = 1\}$ are topologically conjugate.*

Proof. Both maps fold the triangle R along the bisectrix of the angle with vertex $(0, 0, 1)$, and act on the vertices by the rule

$$(1, 0, 0) \mapsto (1, 0, 0), \quad (0, 1, 0) \mapsto (0, 0, 1), \quad (0, 0, 1) \mapsto (1, 0, 0).$$

Both maps are projective (the map F is affine), hence they map straight lines to straight lines.

It is sufficient hence to prove that for every sequence i_1, i_2, i_3, \dots of symbols **1** and **2** the diameters of the nested triangles

$$\Delta_{i_1 i_2 \dots i_n} = \tilde{\Phi}_{i_1} \circ \tilde{\Phi}_{i_2} \circ \dots \circ \tilde{\Phi}_{i_n}(\Delta)$$

exponentially converge to zero as n grows.

The vertices of the triangle $\Delta_{i_1 i_2 \dots i_n}$ are the normalized (i.e., divided by the sum of their coordinates) columns of the matrix

$$B_{i_1 i_2 \dots i_n} = B_{i_1} B_{i_2} \cdots B_{i_n},$$

where

$$B_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $\vec{a}_n, \vec{b}_n, \vec{c}_n$ be first, second, and third columns of the matrix $B_{i_1 i_2 \dots i_n}$, respectively and let a_n, b_n, c_n be the sums of the coordinates of the vectors $\vec{a}_n, \vec{b}_n, \vec{c}_n$.

Lemma 7.3. *For every $n \geq 1$ we have*

$$\frac{1}{3} < \frac{a_n}{c_n} < 3, \quad \frac{1}{3} < \frac{a_n}{b_n} < \frac{3}{2}, \quad \frac{1}{3} < \frac{c_n}{b_n} < \frac{3}{2}.$$

Proof. Let us prove the lemma by induction. We have $(a_1, b_1, c_1) = (1, 1, 1)$, which satisfies the conditions of the lemma.

The sequence $(a_{n+1}, b_{n+1}, c_{n+1})$ is equal to one of the sequences

$$(2c_n, a_n + c_n, b_n), \quad (2a_n, a_n + c_n, b_n).$$

Since a_n and c_n play a symmetric role in our lemma, it is sufficient to check only the second case $a_{n+1} = 2a_n, b_{n+1} = a_n + c_n, c_{n+1} = b_n$.

Then

$$\frac{a_{n+1}}{c_{n+1}} = \frac{2a_n}{b_n} \in \left(\frac{2}{3}, 3\right) \subset \left(\frac{1}{3}, 3\right).$$

We also have

$$\begin{aligned} \frac{a_{n+1}}{b_{n+1}} &= \frac{2a_n}{a_n + c_n} < \frac{2a_n}{a_n + a_n/3} = \frac{3}{2}, \\ \frac{a_{n+1}}{b_{n+1}} &= \frac{2a_n}{a_n + c_n} > \frac{2a_n}{a_n + 3a_n} = \frac{1}{2} > \frac{1}{3}, \end{aligned}$$

and

$$\begin{aligned} \frac{c_{n+1}}{b_{n+1}} &= \frac{b_n}{a_n + c_n} < \frac{b_n}{b_n/3 + b_n/3} = \frac{3}{2}, \\ \frac{c_{n+1}}{b_{n+1}} &= \frac{b_n}{a_n + c_n} > \frac{b_n}{3b_n/2 + 3b_n/2} = \frac{1}{3}, \end{aligned}$$

which finishes the inductive argument. \square

Denote by $\vec{\alpha}_n = \frac{\vec{a}_n}{a_n}, \vec{\beta}_n = \frac{\vec{b}_n}{b_n}$ and $\vec{\gamma}_n = \frac{\vec{c}_n}{c_n}$ the vertices of the triangle $\Delta_{i_1 i_2 \dots i_n}$.

We have that either

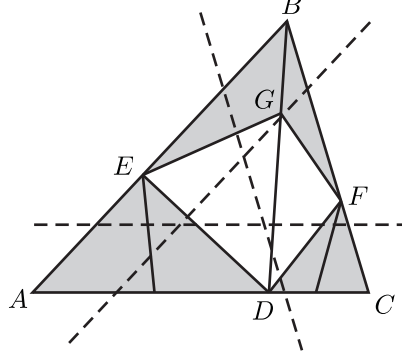
$$\vec{a}_{n+1} = 2\vec{a}_n, \quad \vec{b}_{n+1} = \vec{a}_n + \vec{c}_n, \quad \vec{c}_{n+1} = \vec{b}_n,$$

or

$$\vec{a}_{n+1} = 2\vec{c}_n, \quad \vec{b}_{n+1} = \vec{a}_n + \vec{c}_n, \quad \vec{c}_{n+1} = \vec{b}_n.$$

Consequently, the vertices of the triangle $\Delta_{i_1 i_2 \dots i_{n+1}}$ are equal to

$$\vec{\alpha}_{n+1} = \vec{\alpha}_n, \quad \vec{\beta}_{n+1} = \frac{a_n \vec{\alpha}_n + c_n \vec{\gamma}_n}{a_n + c_n}, \quad \vec{\gamma}_{n+1} = \vec{\beta}_n.$$

FIGURE 13. Triangles $\Delta_{i_1 \dots i_{n+3}}$

The vertices of the triangle $\Delta_{i_1 i_2 \dots i_n 2}$ are equal to

$$\vec{\alpha}_{n+1} = \vec{\gamma}_n, \quad \vec{\beta}_{n+1} = \frac{a_n \vec{\alpha}_n + c_n \vec{\gamma}_n}{a_n + c_n}, \quad \vec{\gamma}_{n+1} = \vec{\beta}_n.$$

Hence, the triangle $\Delta_{i_1 i_2 \dots i_n}$ is divided into the triangles $\Delta_{i_1 i_2 \dots i_n 1}$ and $\Delta_{i_1 i_2 \dots i_n 2}$ by the line connecting the vertex $\vec{\beta}_n$ with the point $\frac{a_n \vec{\alpha}_n + c_n \vec{\gamma}_n}{a_n + c_n}$ on the opposite side of the triangle. Note that by Lemma 7.3

$$\begin{aligned} \left| \vec{\alpha}_n - \frac{a_n \vec{\alpha}_n + c_n \vec{\gamma}_n}{a_n + c_n} \right| &= \frac{c_n}{a_n + c_n} |\vec{\alpha}_n - \vec{\gamma}_n| < \\ &< \frac{c_n}{c_n/3 + c_n} |\vec{\alpha}_n - \vec{\gamma}_n| = \frac{3}{4} |\vec{\alpha}_n - \vec{\gamma}_n|. \end{aligned}$$

Similarly,

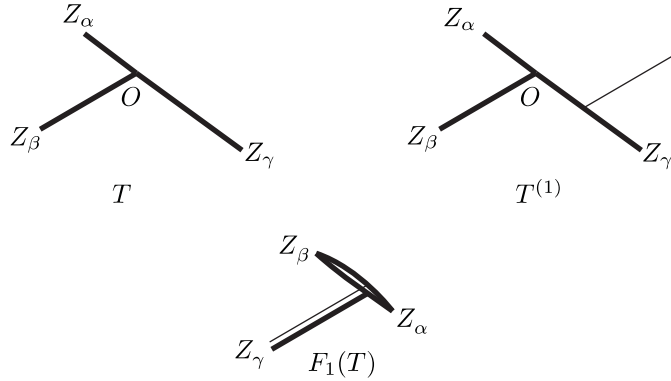
$$\left| \vec{\gamma}_n - \frac{a_n \vec{\alpha}_n + c_n \vec{\gamma}_n}{a_n + c_n} \right| < \frac{3}{4} |\vec{\alpha}_n - \vec{\gamma}_n|.$$

Hence the line dividing $\Delta_{i_1 i_2 \dots i_n}$ into the triangles $\Delta_{i_1 i_2 \dots i_n 1}$ and $\Delta_{i_1 i_2 \dots i_n 2}$ divides the side $[\vec{\alpha}_n, \vec{\gamma}_n]$ in proportion between 1 : 3 and 3 : 1.

Let us prove that diameter of the triangles $\Delta_{i_1 \dots i_{n+1} i_{n+2} i_{n+3}}$ is less than $\frac{3}{4}$ of diameter of the triangle $\Delta_{i_1 i_2 \dots i_n}$. Figure 13 shows how the triangle $\Delta_{i_1 i_2 \dots i_n} = \triangle ABC$ is subdivided into the 8 triangles $\Delta_{i_1 i_2 \dots i_n i_{n+1} i_{n+2} i_{n+3}}$ for different $i_{n+1}, i_{n+2}, i_{n+3}$ (ignore the dashed lines and shading for a while).

By the proved above, the points D , E and F divide the sides of the triangle $\triangle ABC$ in proportion between 1 : 3 to 3 : 1. Similarly the point G divides the segment BD in a proportion belonging to the same range. It follows that the shaded triangles $\triangle AED$, $\triangle CDF$ and quadrilateral $BFGE$ are subsets of the images of the triangle $\triangle ABC$ under the homotheties with coefficient $\frac{3}{4}$ and centers in the points A , C , and B , respectively. The images of the lines BC , AB , and AC under these homotheties are shown as dashed lines on Figure 13.

It follows that diameters of the shaded triangles and quadrilateral are less than three quarters of the diameter of $\triangle ABC$. The diagonal GD of the quadrilateral $EDFG$ is less than $\frac{3}{4}$ times the length of the segment BD . The other diagonal and the sides of $EDFG$ belong to one of the shaded triangles or quadrilateral, hence their lengths are also less than three quarters of the diameter of $\triangle ABC$.

FIGURE 14. Extension of F_1 to a covering

Consequently, diameter of the quadrilateral $EDFG$ is also less than $\frac{3}{4}$ times the diameter of $\triangle ABC$.

Each of the triangles $\Delta_{i_1 \dots i_{n+1} i_{n+2} i_{n+3}}$ is a subset of one of the triangles and quadrilaterals for which we have proved that their diameter is less than $\frac{3}{4}$ times the diameter of $\triangle ABC = \Delta_{i_1 i_2 \dots i_n}$, which finishes the proof. \square

Definition 7.2. Let (k_1, k_2, \dots) be the F_1 -itinerary of the tripod T . Denote by $T^{(n)}$ the tree $\Phi_{k_1} \circ \Phi_{k_2} \circ \dots \circ \Phi_{k_n}(F_1^n(T))$.

It follows from the description of the transformations F_1 and Φ_i that the Hubbard tripod of $T^{(n)}$ is isometric to the tripod T . In particular, T is naturally identified with the Hubbard tripod of $T^{(1)} = \Phi_{k_1}(F_1(T))$ (see Figure 14).

The tree $T^{(1)}$ is obtained from T by attaching a copy of the segment OZ_β to the point of the geodesic $Z_\alpha Z_\gamma$ symmetric to O with respect to the center of $Z_\alpha Z_\gamma$, i.e., by making the tree symmetric with respect to the midpoint of $Z_\alpha Z_\gamma$. The covering $T^{(1)} \rightarrow F_1(T)$ folds then the segment $Z_\alpha Z_\gamma$ twice and identifies the leg OZ_β with the added copy of it.

The midpoint of $Z_\alpha Z_\gamma$ divides $T^{(1)}$ into two copies of $F_1(T)$ such that the covering $T^{(1)} \rightarrow F_1(T)$ is an isometry of these copies with $F_1(T)$.

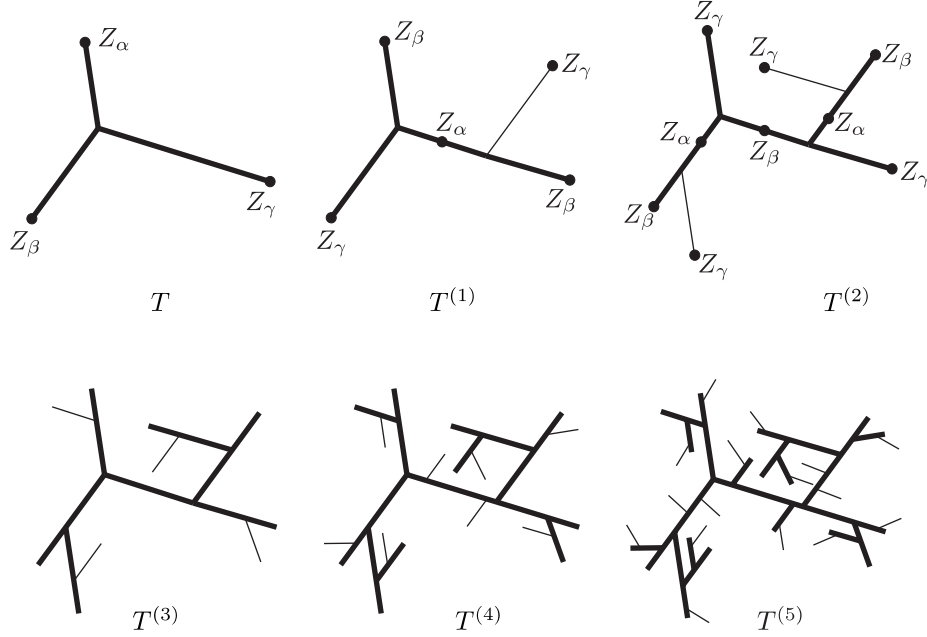
By the description of the transformations Φ_i , the tree $T^{(n+1)}$ is obtained then as two copies of $F_1(T)^{(n)}$ pasted together along the copies of the vertex Z_α of $F_1(T)$, which will be the midpoint of $Z_\alpha Z_\gamma$ in $T^{(n+1)}$. This shows by induction that the tree $T^{(n)}$ is naturally identified with a subtree of $T^{(n+1)}$ and that the map $F_1 : T \rightarrow F_1(T)$ is extended to a branched two-to-one coverings $T^{(n+1)} \rightarrow F_1(T)^{(n)}$.

We get then for every n a branched covering $T^{(n)} \rightarrow F_1^n(T)$ of degree 2^n equal to the composition of the degree two coverings

$$T^{(n)} \rightarrow F_1(T)^{(n-1)} \rightarrow F_1^2(T)^{(n-2)} \rightarrow \dots \rightarrow F_1^n(T).$$

Consequently, $T^{(n)}$ consists of 2^n copies of the tripod $F_1^n(T)$ connected to each other along their feet in some way. Then the tree $T^{(n+1)}$ is obtained from $T^{(n)}$ by making each of these copies of $F_1^n(T)$ symmetric with respect to the midpoint of the copy of the segment $Z_\alpha Z_\gamma$ of $F_1^n(T)$.

See on Figure 15 the sequence $T^{(n)}$, $n = 0, \dots, 5$ for a concrete tripod T . On the first three trees $T^{(k)}$ the labels come from the labelling of the copies of the tripod

FIGURE 15. Growing the trees $T^{(n)}$

$F_1^k(T)$, i.e., the labels of the images of the corresponding points under the covering $F_1^k : T^{(k)} \rightarrow F_1^k(T)$. In each of the trees $T^{(n)}$ the sub-tree $T^{(n-1)}$ is shown by thicker lines.

Proposition 7.4. *Let d_n be the Hausdorff distance between $T^{(n)}$ and $T^{(n+1)}$ inside $T^{(n+1)}$, i.e., maximum over $\xi \in T^{(n+1)}$ of the distance of ξ to $T^{(n)}$. Then $d_n < \frac{d}{2^{(n-1)/2}}$, where d is mass of T .*

Proof. The tree $T^{(n+1)}$ is obtained from $T^{(n)}$ by making each of the 2^n copies of $F_1^n(T)$ in $T^{(n)}$ symmetric with respect to the midpoint of $Z_\alpha Z_\gamma$, i.e., by attaching a copy of the segment OZ_β of $F_1^n(T)$. It follows that the Hausdorff distance between $T^{(n)}$ and $T^{(n+1)}$ is equal to the length of the segment OZ_β in $F_1^n(T)$. But it is not more than mass of $F_1^n(T)$, which is not more than $\frac{d}{2^{(n-1)/2}}$, by Lemma 7.1. \square

Definition 7.3. Let T be a tripod. Then the *limit dendrite* of T , denoted $T^{(\infty)}$ is completion of the inductive limit of the metric trees $T^{(n)}$ as n goes to infinity.

Denote by $\tilde{\iota}_n : T^{(n+1)} \rightarrow T^{(n)}$ the projection of $T^{(n+1)}$ onto its subtree $T^{(n)}$.

Proposition 7.5. *The limit dendrite $T^{(\infty)}$ is homeomorphic to the inverse limit of the trees $T^{(n)}$ with respect to the maps $\tilde{\iota}_n$.*

Proof. It follows directly from Proposition 7.4 that the inductive limit of the spaces $T^{(n)}$ is completely bounded (i.e., has a finite ϵ -net for every positive ϵ). It follows then that the completion $\mathcal{D}(T)$ is compact.

Note that the projections $\tilde{\iota}_n$ are idempotent maps (i.e., $\tilde{\iota}_n(x) = x$ for all $x \in T^{(n)}$). By Proposition 7.4, the map $\tilde{\iota}_n$ moves points not more than by $\frac{d}{2^{(n-1)/2}}$.

Let $x_n \in T^{(n)}$ be a sequence representing a point x of the inverse limit, i.e., such that $\tilde{\iota}_n(x_{n+1}) = x_n$. It follows from Proposition 7.4 that the corresponding sequence $x_n \in T^{(\infty)}$ is fundamental, hence it converges to a point $\delta(x)$ of $T^{(\infty)}$.

Let us show that the map δ from the inverse limit to $T^{(\infty)}$ is a homeomorphism. It is continuous by Proposition 7.4. If x_n and y_n are sequences representing different points of the inverse limit, then $d(x_k, y_k) > 0$ for some k , which implies that $d(x_n, y_m) \geq d(x_k, y_k)$ for all $n, m \geq k$, by the elementary properties of trees and projections $\tilde{\iota}_n$. It follows that the map δ is injective.

Let $x \in T^{(\infty)}$ be arbitrary. It is a limit of a fundamental sequence x_n in the inductive limit of the trees $T^{(n)}$. Passing to a subsequence, and then repeating the entries of the subsequence, if necessary, we may assume that $x_n \in T^{(n)}$. Consider for every k the sequence $x_{k,n} = \tilde{\iota}_k \circ \dots \circ \tilde{\iota}_{n-1}(x_n)$ for $n > k$. As above, we have $d(x_n, x_m) \geq d(x_{k,n}, x_{k,m})$ for all $m, n > k$, which implies that the sequence $x_{k,n}$ converges to a point y_k in $T^{(k)}$. We have $\tilde{\iota}(y_{k+1}) = y_k$. Limit of the sequence y_n is equal to the limit of x_n , since $\tilde{\iota}_n$ moves points not more than by $\frac{d}{2^{(n-1)/2}}$. Consequently, δ is onto, and hence a homeomorphism (since the inverse limit and $T^{(\infty)}$ are compact). \square

Theorem 7.6. *Denote by $\psi : \Delta \rightarrow \Delta$ the homeomorphism conjugating the maps F and \tilde{F} , i.e., such a homeomorphism that $\psi \circ F = \tilde{F} \circ \psi$.*

If the image of $\zeta \in \mathcal{S}_\infty$ in \mathcal{S}_0 is ζ_0 and (x_1, x_2, x_3) are the lengths of legs of the tripod $\rho^{-1}(\zeta_0)$, then denote by T_ζ the tripod with the lengths of legs $\psi(x_1, x_2, x_3)$.

Then there exists a family of homeomorphisms $\tau_\zeta : T_\zeta^{(\infty)} \rightarrow \rho_\infty^{-1}(\zeta)$ conjugating the maps $p_\infty : \rho_\infty^{-1}(\zeta) \rightarrow \rho_\infty^{-1}(q_\infty(\zeta))$ with the coverings $\tilde{F} : T_\zeta^{(\infty)} \rightarrow T_{q_\infty(\zeta)}^{(\infty)}$.

Here \tilde{F} is covering of limit dendrites obtained as the limit of the coverings $T^{(n)} \rightarrow \tilde{F}_1(T)^{(n-1)}$.

Proof. Let $(\zeta_0, \zeta_1 = \lambda^{-1}(\zeta_0), \zeta_2 = \lambda_1^{-1}(\zeta_1), \dots)$ be a point of \mathcal{S}_∞ . Let (x_1, x_2, x_3) be the lengths of legs of the tripod $\rho^{-1}(\zeta_0)$. Let (k_1, k_2, \dots) be the itinerary of the tripod $\rho^{-1}(\zeta_0)$ (with respect to the folding map F).

Then (k_1, k_2, \dots) is the F_1 -itinerary of the tripod T_ζ with the lengths of legs $\psi(x_1, x_2, x_3)$. The triple of lengths of legs of the tripod $F_1^n(T_\zeta)$ is $F_1^n(\psi(x_1, x_2, x_3))$. If we divide the triple $F_1^n(\psi(x_1, x_2, x_3))$ by the sum of its entries, then we get $\tilde{F}^n(\psi(x_1, x_2, x_3)) = \psi(F^n(x_1, x_2, x_3))$.

It follows that the tripod T_n with the lengths of legs $F^n(x_1, x_2, x_3)$ is homeomorphic to the tripod $F_1^n(T_\zeta)$. Let us fix some homeomorphism $F_1^n(T_\zeta) \rightarrow T_n$. Applying it to the copies of $F_1^n(T_\zeta)$ in $T_\zeta^{(n)} = \Phi_{k_1} \circ \Phi_{k_2} \circ \dots \circ \Phi_{k_n}(F_1^n(T_\zeta))$, we get a homeomorphism

$$\tau_n : T_\zeta^{(n)} \rightarrow \rho_n^{-1}(\zeta_n),$$

since the tree $\rho_n^{-1}(\zeta_n)$ is homeomorphic to $\Phi_{k_1} \circ \Phi_{k_2} \circ \dots \circ \Phi_{k_n}(T_n)$ by Proposition 6.6.

It follows from the description of the action of $\iota_n : \rho_{n+1}^{-1}(\zeta_{n+1}) \rightarrow \rho_n^{-1}(\zeta_n)$ given in Proposition 6.6 that the homeomorphisms τ_n conjugate ι_n with the projection $\tilde{\iota}_n : T_\zeta^{(n+1)} \rightarrow T_\zeta^{(n)}$. Therefore, the limit of the homeomorphisms τ_n is a homeomorphism $\tau_\zeta : T_\zeta^{(\infty)} \rightarrow \rho_\infty^{-1}(\zeta)$.

The statement about the map p_n follows now from Proposition 6.3. \square

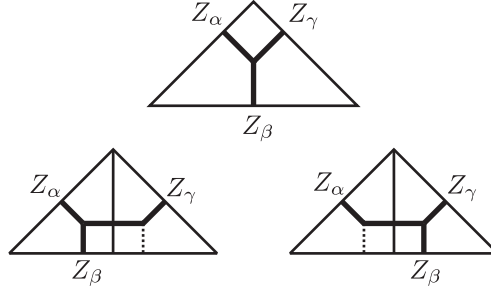


FIGURE 16. Unfolding the tripods inside of a triangle

We get in this way a family of length metrics on the slices of the Julia set of f such that for every w_0 there exists a constant $c \geq 1$ such that the map $f : J_{w_0} \rightarrow J_{(1-2/w_0)^2}$ multiplies every curve of J_{w_0} by c .

7.2. Triangle-filling. The inductive unfolding procedures Φ_i , described in Subsection 6.3 can be realized in a way leading to a family of surjections from the dendrites $T^{(\infty)}$ (i.e., from the slices of the Julia set of f) onto an isosceles right triangle.

Draw a tripod $T = Z_\alpha Z_\beta Z_\gamma$ inside an isosceles right triangle in such a way that its foot Z_β belongs to the hypotenuse and the feet Z_α and Z_γ are symmetric points on the catheti, see top of Figure 16.

We can now paste two copies of the tripod T together with the circumscribed triangle in such a way that the copies are reflections of each other and the union of the two triangles is again an isosceles right triangle.

If we label three vertices of tree $\Phi_i(T)$, accordingly to the unfolding rule, the vertices Z_α and Z_γ will be again symmetric points on the catheti and the vertex Z_β will belong to the hypotenuse. See bottom of Figure 16, where both case I (on the left hand side) and case II (on the right hand side) are shown.

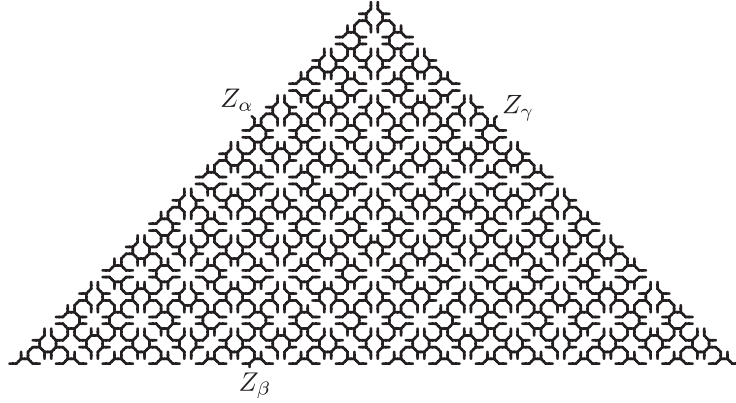
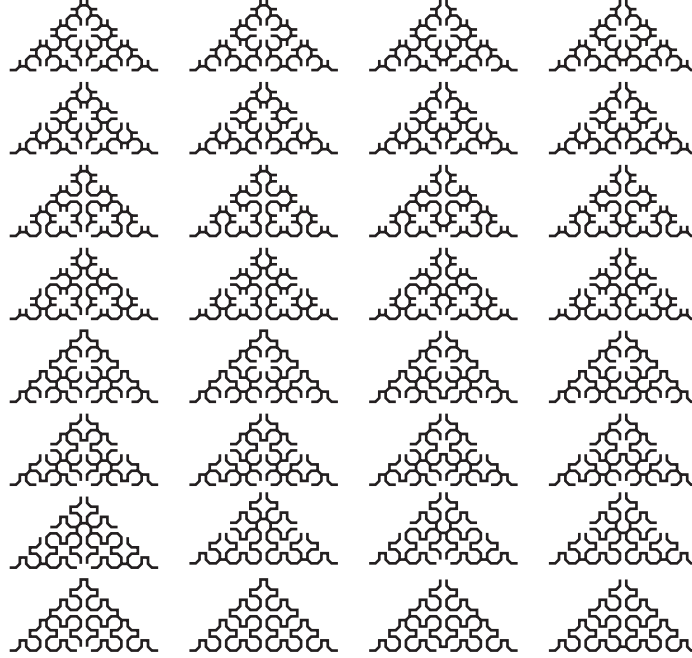
We can iterate now the process (choosing one of the two cases on each step). For better visualization (in order the vertices of the trees $\Phi_{i_1} \circ \dots \circ \Phi_{i_n}(T)$ not to collide), we may on each step shorten (or delete) the edge containing the copy of the vertex Z_γ of $\Phi_{i_2} \circ \dots \circ \Phi_{i_n}(T)$ that did not become the vertex Z_β of $\Phi_{i_1} \circ \dots \circ \Phi_{i_n}(T)$. In this case only three vertices $Z_\alpha, Z_\beta, Z_\gamma$ of $\Phi_{i_1} \circ \dots \circ \Phi_{i_n}(T)$ will belong to the perimeter of the triangle.

See, for instance Figure 17, where the result of application of this procedure ten times is shown. Here always the transformation Φ_1 is applied and we have deleted the edges containing the unlabeled copies of Z_γ . We get in this way the graph of the action of $\text{IMG}(z^2 + i)$ on the tenth level of the tree, approximating the Julia set of $z^2 + i$.

If we apply Φ_2 each time, then we get in the limit the well known *Sierpiński plane filling curve*, see [Sie12, Man82].

Figure 18 shows all possible graphs obtained in this way by applying seven transformations Φ_i .

7.3. External rays. A standard tool in dynamics of complex polynomials are *external rays* (see [DH84, DH85]). If K is the *filled-in Julia set* of a monic polynomial f (K is equal to the Julia set, if it is a dendrite), then there exists a bi-holomorphic

FIGURE 17. A Schreier graph of $\text{IMG}(z^2 + i)$ FIGURE 18. Schreier graphs of Γ_0

isomorphism of the complement of the unit disc $\mathbb{C} \setminus \mathbb{D}$ with $\mathbb{C} \setminus K$, conjugating the action of $z \mapsto z^{\deg f}$ on $\mathbb{C} \setminus \mathbb{D}$ with the action of f on $\mathbb{C} \setminus K$. The images R_θ of the rays $\{r \exp i\theta : r > 1\} \subset \mathbb{C} \setminus \mathbb{D}$ in $\mathbb{C} \setminus K$ under this isomorphism are called the *external rays* to the Julia set of f .

Since our rational function $f(z, w) = \left(\left(1 - \frac{2z}{w}\right)^2, \left(1 - \frac{2}{w}\right)^2 \right)$ is a polynomial on the first coordinate, we also may define external rays to the Julia set of f .

Namely, for every $w_0 \in \widehat{\mathbb{C}}$ let J_{w_0} be the intersection of the Julia set of f with the line $w = w_0$. We know that J_{w_0} is a dendrite. The complement of J_{w_0} in

the line $w = w_0$ (without the superattracting point $[1 : 0 : 0]$) is an annulus, bi-holomorphically isomorphic to the complement of a disc in \mathbb{C} . We define the rays R_{θ, w_0} in the line $w = w_0$ in the same way as for complex polynomials. The angle θ of a ray is not canonically defined, since the bi-holomorphic isomorphism is defined up to a rotation. The set of all external rays R_{θ, w_0} is in a bijective correspondence with the boundary of a small ball around $[1 : 0 : 0]$, i.e., with a 3-sphere. We introduce a topology on the set \mathcal{R} of external rays $R_{\theta, w}$ using this bijection. The function f induces then a self-map $f_{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{R}$, since the image of a ray $R_{\theta, w}$ under the action of f will be a ray $R_{\theta', \hat{f}}$. The aim of this section is to understand the action of $f_{\mathcal{R}}$ on \mathcal{R} .

The action of f on \mathcal{R} is expanding (i.e., sub-hyperbolic), since it is a double covering of the circle of rays corresponding to $(1 - 2/w)^2$ by the circle of rays corresponding to w , and the rational function $(1 - 2/w)^2$ is sub-hyperbolic. It follows that the dynamical system $(\mathcal{R}, f_{\mathcal{R}})$ is topologically conjugate to the limit dynamical system of the iterated monodromy group of $f_{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{R}$. This group is the subgroup of $\text{IMG}(f)$ generated by the loops not intersecting the Julia set of f , i.e.,

$$\text{IMG}(f_{\mathcal{R}}) = \langle \alpha\gamma\beta, s, t \rangle.$$

By Theorem 4.3, we have

$$\alpha\gamma\beta = \sigma(\beta, 1, 1, \beta)(\alpha, \gamma, \alpha, \beta\gamma\beta) = \sigma(\beta\alpha, \gamma, \alpha, \gamma\beta).$$

Compose the wreath recursion with conjugation by $(\alpha, \beta, \alpha, 1)$. We get then in the new wreath recursion:

$$\begin{aligned} \alpha\gamma\beta &= \sigma(\beta \cdot \beta\alpha \cdot \alpha, \alpha \cdot \gamma \cdot \beta, \alpha \cdot \alpha, \alpha \cdot \gamma\beta) = \sigma(1, \alpha\gamma\beta, 1, \alpha\beta\gamma), \\ s &= \pi(\alpha\alpha, \beta\beta, \alpha\alpha, \beta\beta) = \pi, \end{aligned}$$

and

$$t = (r^\alpha, r^\beta, t^\alpha, t).$$

We have, by relations (1)–(3)

$$\begin{aligned} r^\alpha &= \alpha\beta\alpha\beta\gamma\beta t^{-1} s^{-1} \alpha = \\ &= t^{-1} \alpha \cdot \gamma\beta\gamma \cdot \alpha \cdot \gamma\beta\gamma \cdot \gamma\beta s^{-1} \alpha = t^{-1} \alpha\gamma\beta\gamma\alpha\gamma s^{-1} \alpha = \\ &= t^{-1} s^{-1} \alpha\gamma \cdot \beta \cdot \alpha\gamma\alpha \cdot \alpha\gamma\alpha = t^{-1} s^{-1} \alpha\gamma\beta, \end{aligned}$$

and $r^\beta = r^{(\alpha\beta)\beta} = r^\alpha$.

Denote $\alpha\gamma\beta = \tau$. We see that the subgroup generated by τ, s, t is self-similar and is given by the recursion

$$(17) \quad \tau = \sigma(1, \tau, 1, \tau),$$

$$(18) \quad s = \pi,$$

$$(19) \quad t = (t^{-1} s^{-1} \tau, t^{-1} s^{-1} \tau, t, t).$$

Note that τ commutes with s and t . We also have $s^2 = 1$ and

$$(st)^2 = (s^{-1} \tau, s^{-1} \tau, t^{-1} s^{-1} t \tau, t^{-1} s^{-1} t \tau),$$

hence

$$(st)^4 = (\tau^2, \tau^2, \tau^2, \tau^2) = \tau^4,$$

and

$$t^4 = (1, 1, t^4, t^4) = 1.$$

Consider the elements $X = t^2s$ and $Y = tst$. We have

$$X = \pi(t^2, t^2, t^{-1}st^{-1}s\tau^2, t^{-1}st^{-1}s\tau^2)$$

and

$$Y = \pi(s\tau, s\tau, t^{-1}st\tau, t^{-1}st\tau).$$

Composing the wreath recursion with conjugation by $(s, s, 1, 1)$, we get the recursion

$$\begin{aligned} X &= \pi(X, X, Y\tau^{-2}, Y\tau^{-2}), \\ Y &= \pi(\tau, \tau, X^{-1}Y\tau, X^{-1}Y\tau), \\ \tau &= \sigma(1, \tau, 1, \tau). \end{aligned}$$

Proposition 7.7. *The group generated by X, Y, τ is isomorphic to the group generated by the matrices $\varphi(X) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \varphi(Y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \varphi(\tau) =$*

$\begin{pmatrix} 1 & 0 & 1/4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. In particular, the group generated by X, Y, τ^4 is isomorphic to the Heisenberg group over integers.

Proof. Note that we have

$$[X, Y] = st^2t^{-1}st^{-1}t^2stst = (st)^4 = \tau^4.$$

It follows that the group $\langle X, Y, \tau \rangle$ is a homomorphic image of the group given by the presentation

$$G_1 = \langle X, Y, \tau : [X, Y] = \tau^4, [X, \tau] = [Y, \tau] = 1 \rangle.$$

Let us prove that it is isomorphic to this group, which will finish the proof, since the group generated by the matrices listed in the proposition is given by this presentation.

Consider the homomorphism $\{1, 2, 3, 4\}^* \rightarrow \{\mathbf{1}, \mathbf{2}\}^*$ of the free monoids given by $1 \mapsto \mathbf{1}, 2 \mapsto \mathbf{1}, 3 \mapsto \mathbf{2}, 4 \mapsto \mathbf{2}$. It is easy to see that this homomorphism agrees with the action of the group $G_0 = \langle X, Y, \tau \rangle$ on the tree $\{1, 2, 3, 4\}$, so that after taking projection we get an action of G_0 on the tree $\{\mathbf{1}, \mathbf{2}\}^*$. This action is given by the wreath recursion

$$X = \sigma(X, Y), \quad Y = \sigma(1, X^{-1}Y), \quad \tau = 1.$$

We get hence an epimorphism from G_0 to the given group acting on the binary tree. It follows from the recursion (see [NS04] and [Nek05, Proposition 2.9.2]) that X and Y generated in a free abelian group in this epimorphic image. Consequently, the kernel of the natural epimorphism $G_1 \rightarrow G_0$ is contained in $\langle \tau \rangle$. But the group $\langle \tau \rangle$ acts faithfully on the tree $\{1, 2, 3, 4\}^*$, hence the kernel is trivial. \square

It is checked directly that the map

$$\phi \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{a-b}{2} & \frac{c}{2} + \frac{a^2-b^2}{8} - \frac{ab}{4} - \frac{a}{2} \\ 0 & 1 & \frac{a+b}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

is an automorphism of the Heisenberg group.

The first level stabilizer of the group $\langle X, Y, \tau \rangle$ is generated (since X and Y commute with τ and $YX = XY\tau^{-4}$) by τ^2, XY, Y^2 , which are equal to

$$\begin{aligned}\tau^2 &= (\tau, \tau, \tau, \tau), \\ XY &= (Y\tau^{-1}, Y\tau^{-1}, Y\tau, Y\tau), \\ Y^2 &= (X^{-1}Y\tau^2, X^{-1}Y\tau^2, X^{-1}Y\tau^2, X^{-1}Y\tau^2).\end{aligned}$$

Direct computations show that the isomorphism φ from Proposition 7.7 conjugates the virtual endomorphism associated with the first coordinate of the wreath recursion with the action of ϕ on the lattice generated by $\varphi(X), \varphi(Y)$, and $\varphi(\tau)$. This gives another proof of Proposition 7.7, but it also gives a description of the limit space of the group $\langle X, Y, \tau \rangle$.

Proposition 7.8. *The limit G -space $\mathcal{X}_{\langle X, Y, \tau \rangle}$ of the group $\langle X, Y, \tau \rangle$ is the real Heisenberg group with the right action of $\langle X, Y, \tau \rangle$ given by the isomorphism φ from Proposition 7.7. The limit space $\mathcal{J}_{\langle X, Y, \tau \rangle}$ is hence the quotient of the Heisenberg group by the action of the lattice $\langle \varphi(X), \varphi(Y), \varphi(\tau) \rangle$.*

The next theorem gives us a description of the action of the map f on the space of the external rays, i.e., the action of f on the neighborhood of “infinity”, i.e., of the superattracting point $[1 : 0 : 0]$. It would be interesting to deduce Theorem 7.9 analytically.

Theorem 7.9. *The subgroup $\langle X, Y, \tau \rangle$ has index 4 in $\langle s, t, \tau \rangle$. The limit G -space $\mathcal{X}_{\text{IMG}(f_{\mathcal{R}})}$ of $\text{IMG}(f_{\mathcal{R}}) = \langle s, t, \tau \rangle$ is the real Heisenberg group together with the action of the group given by*

$$\begin{aligned}\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot s &= \begin{pmatrix} 1 & -x-1 & z+2y \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot t &= \begin{pmatrix} 1 & -y-1 & z-xy-x-1 \\ 0 & 1 & x+1 \\ 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

and

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \tau = \begin{pmatrix} 1 & x & z+1/4 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The space of external rays \mathcal{R} of the function f is homeomorphic to the quotient of the real Heisenberg group by the described action. The action $f_{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{R}$ of f on it is induced by the automorphism

$$\phi^{-1} : \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a+b & 2c+a+b-ab-\frac{a^2-b^2}{2} \\ 0 & 1 & b-a \\ 0 & 0 & 1 \end{pmatrix}$$

on the Heisenberg group.

Note that the automorphism ϕ^{-1} coincides (up to an inner automorphism of the real Heisenberg group) with the automorphism used in [Gel94], see also [PN08].

Proof. The first level stabilizer of the group $\langle s, t, \tau \rangle$ is generated by τ^2, t, sts . They are equal (in the wreath recursion (17)–(19) conjugated by $(s, s, 1, 1)$) to

$$\begin{aligned}\tau^2 &= (\tau, \tau, \tau, \tau), \\ t &= (st^{-1}\tau, st^{-1}\tau, t, t), \\ sts &= (sts, sts, t^{-1}s\tau, t^{-1}s\tau).\end{aligned}$$

The following equalities are checked directly:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot sts = \begin{pmatrix} 1 & -y & z - xy - x + y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot st^{-1}\tau = \begin{pmatrix} 1 & -1-y & z - xy - x + y + 1/4 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned}\phi\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot sts\right) &= \begin{pmatrix} 1 & \frac{-x-y}{2} & \frac{z-xy-x+y}{2} + \frac{y^2-x^2}{8} + \frac{xy}{4} + \frac{y}{2} \\ 0 & 1 & \frac{x-y}{2} \\ 0 & 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & \frac{-x-y}{2} & \frac{z}{2} + \frac{y^2-x^2}{8} - \frac{xy}{4} - \frac{x}{2} + y \\ 0 & 1 & \frac{x-y}{2} \\ 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}\phi\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot sts\right) &= \begin{pmatrix} 1 & \frac{x-y}{2} & \frac{z}{2} + \frac{x^2-y^2}{8} - \frac{xy}{4} - \frac{x}{2} \\ 0 & 1 & \frac{x+y}{2} \\ 0 & 0 & 1 \end{pmatrix} \cdot sts = \\ &= \begin{pmatrix} 1 & \frac{-x-y}{2} & \frac{z}{2} + \frac{y^2-x^2}{8} - \frac{xy}{4} - \frac{x}{2} + y \\ 0 & 1 & \frac{x-y}{2} \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

$$\begin{aligned}\phi\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot t\right) &= \\ &= \begin{pmatrix} 1 & \frac{-x-y}{2} - 1 & \frac{z-xy-x-1}{2} + \frac{(1+y)^2-(x+1)^2}{8} + \frac{(x+1)(y+1)}{4} + \frac{1+y}{2} \\ 0 & 1 & \frac{x-y}{2} \\ 0 & 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} 1 & \frac{-x-y}{2} - 1 & \frac{z}{2} - \frac{xy}{4} + \frac{y^2-x^2}{8} - \frac{x}{2} + y + \frac{1}{4} \\ 0 & 1 & \frac{x-y}{2} \\ 0 & 0 & 1 \end{pmatrix},\end{aligned}$$

while

$$\phi \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot st^{-1}\tau = \begin{pmatrix} 1 & \frac{x-y}{2} & \frac{z}{2} + \frac{x^2-y^2}{8} - \frac{xy}{4} - \frac{x}{2} \\ 0 & 1 & \frac{x+y}{2} \\ 0 & 0 & 1 \end{pmatrix} \cdot st^{-1}\tau =$$

$$\begin{pmatrix} 1 & \frac{-x-y}{2} - 1 & \frac{z}{2} - \frac{xy}{4} + \frac{y^2-x^2}{8} - \frac{x}{2} + y + \frac{1}{4} \\ 0 & 1 & \frac{x-y}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

We see that in both cases ϕ conjugates the transformations to the action of the first coordinate of the wreath recursion, which finishes the proof. \square

7.4. Smooth cases. Let A', B', C' be the vertices of the triangle $\mathcal{D} \subset \mathcal{S}$, as in Subsection 6.1. Note that $A' \cdot a = A'$, $B' \cdot a = B'$, and $C' \cdot a = C'$ in \mathcal{S} , since these points belong to the boundary of \mathcal{D} , which is identified in \mathcal{S} with the boundary of $\mathcal{D} \cdot a$. Note also that since $L(A' \otimes 2) = A'$, the point A' is invariant under the map $q \circ \lambda^{-1} : \mathcal{S} \rightarrow \mathcal{S}$. Consequently, the fiber $\rho^{-1}(A')$ is invariant under the map $p \circ \theta : \mathcal{M} \rightarrow \mathcal{M}$.

The set $\rho^{-1}(A')$ is the tripod with the lengths of legs $(1, 0, 0)$, so that the points Z_γ and Z_β coincide, i.e., it is a segment with the ends Z_α and $Z_\gamma = Z_\beta$. The map $p \circ \theta : \rho^{-1}(A') \rightarrow \rho^{-1}(A')$ doubles the distances inside the segment $\rho^{-1}(A')$ and then folds it in two. The common image of the endpoints Z_α and $Z_\gamma = Z_\beta$ is $Z_\beta = Z_\gamma$, the image the midpoint is Z_α (and the image of $Z_\beta = Z_\gamma$ is $Z_\gamma = Z_\beta$), see Subsection 6.4. Hence, the map $p \circ \theta : \rho^{-1}(A') \rightarrow \rho^{-1}(A')$ is the tent map.

The itinerary of the tripod $\rho^{-1}(A')$ is therefore $(\mathbf{2}, \mathbf{2}, \mathbf{2}, \dots)$. Consequently, the fibers $\rho_n^{-1}(A')$ and $\rho_\infty^{-1}(A')$ are also segments, and the maps $\iota_n : \rho_{n+1}^{-1}(A') \rightarrow \rho_n^{-1}(A')$ are homeomorphisms.

The corresponding slice of the Julia set of f is J_{w_0} for $w_0 = 1$, where the map $f : J_1 \rightarrow J_1$ is equal to the polynomial $z \mapsto (1 - 2z)^2$. This is the Ulam-von Neumann map (see, for instance [Mil99, Section 7, Example 2]), its Julia set is the interval $[0, 1]$, and it is topologically conjugate on the Julia set with the tent map.

Since the extension of F onto the limit dendrites are branched coverings, we immediately get a set of limit dendrites that are graphs.

Proposition 7.10. *The intersection J_{w_0} of the Julia set of f with the line $w = w_0$ is homeomorphic to a finite tree if and only if w_0 is mapped onto the fixed point 1 by some iteration of $\hat{f}(w) = (1 - 2/w)^2$.*

Proof. Let $\zeta = (\zeta_0, \zeta_1, \dots) \in \mathcal{S}_\infty$, where $\zeta_{n+1} = \lambda_n^{-1}(\zeta_n)$. Let (x_1, x_2, x_3) be the lengths of legs of the tripod $\rho^{-1}(\zeta_0)$. We have to show that $\rho_\infty^{-1}(\zeta)$ is a finite tree if and only if the point ζ is mapped onto A' by some iteration of the map $q \circ \lambda^{-1}$, i.e., that $F^n(x_1, x_2, x_3) = (1, 0, 0)$ for some n .

By Proposition 6.6, the tree $\rho_n^{-1}(\zeta_n)$ is obtained by pasting together 2^n copies of the tripod T_n with the lengths of legs $F^n(x_1, x_2, x_3)$. If T_n has all legs of positive length, then $\rho_n^{-1}(\zeta_n)$ has at least 2^n vertices of degree 3 (the copies of the common point of the legs). If T_n has one leg of length zero, but all three feet are different, then the Hubbard tripod of $\Phi_{k_{n-2}} \circ \Phi_{k_{n-1}} \circ \Phi_{k_n}(F^n(T))$ has all legs of positive lengths for any triple of indices k_n, k_{n-1}, k_{n-2} , hence the tree $\rho_n^{-1}(\zeta_n)$ has at least 2^{n-3} vertices of degree 3.

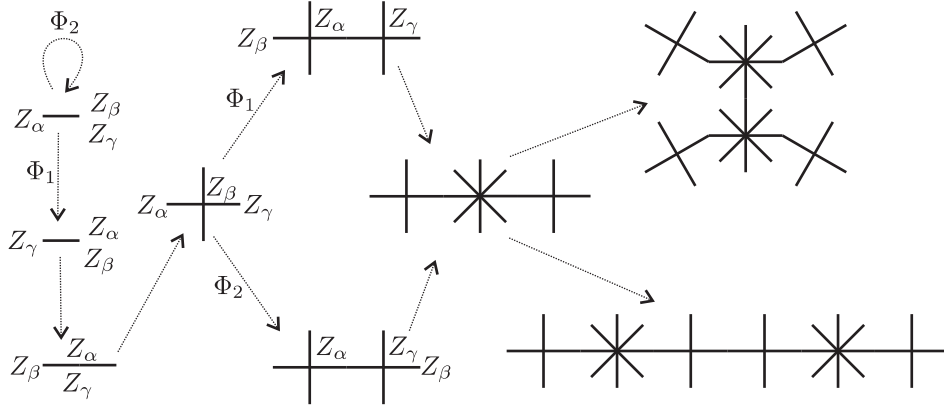
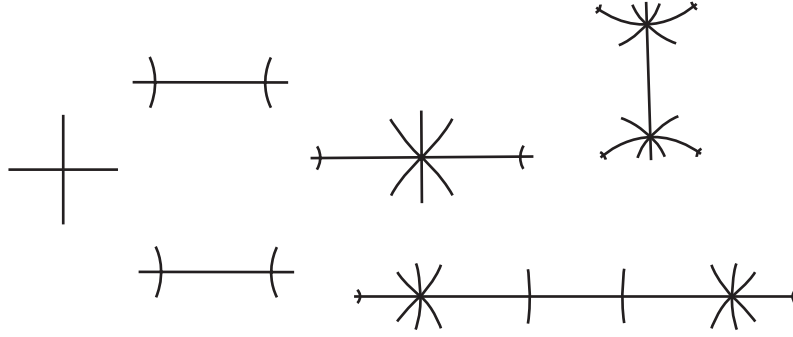
FIGURE 19. Dendrites $\rho_\infty^{-1}(\zeta)$ that are graphs

FIGURE 20. Piecewise smooth slices of the Julia set

In any case, the number of vertices of $\rho_n^{-1}(\zeta_n)$ goes to infinity and $\rho_\infty^{-1}(\zeta)$ can not be a finite graph, if $F^n(x_1, x_2, x_3)$ never has less than two non-zero coordinates. Note that $F(0, 1, 0) = (0, 0, 1)$ and $F(0, 0, 1) = (1, 0, 0)$, which implies that if $F^n(x_1, x_2, x_3)$ has only one non-zero coordinate, then $F^{n+2}(x_1, x_2, x_3) = (1, 0, 0)$.

If, on the other hand, $F^n(x_1, x_2, x_3) = (1, 0, 0)$, then $F^m(x_1, x_2, x_3) = (1, 0, 0)$ for all $m \geq n$, and all graphs $\rho_m^{-1}(\rho_m)$ are isomorphic for $m \geq n$ (with respect to the homeomorphisms $\iota_m : \rho_{m+1}^{-1}(\rho_{m+1}) \rightarrow \rho_m^{-1}(\rho_m)$) and are obtained by gluing together a finite number of segments, i.e., are finite simplicial graphs. \square

See the first six generations of the graphs $\Phi_{i_1} \circ \dots \circ \Phi_{i_k}(\rho^{-1}(A'))$ on Figure 19. The corresponding slices of the Julia set of the rational map f are shown on Figure 20.

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